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SPONSORED PROJECT INITIATION

Date: March 19, 1979

Project Title: Expectations and Equilibrium over Time

Project No: M-50-632 Green card

Project Director: Dr. David C. Nachman

Sponsor: National Science Foundation

Agreement Period: From 2/1/79 Until 7/31/81*
*Includes 6 month flexibility period

Type Agreement: Grant No. SOC-7820169

Amount:
$66,802 NSF (M-50-632)
5,922 GIT Contribution (M-50-321)
$72,724 Total

Reports Required: Annual Progress Reports; Final Project Report

Sponsor Contact Person (s):

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Defense Priority Rating: none

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SPONSORED PROJECT TERMINATION SHEET

Date 11/11/81

Project Title: Expectations and Equilibrium over Time

Project No: M-50-632

Project Director: Dr. David C. Nachman

Sponsor: National Science Foundation

Effective Termination Date: 7/31/81

Clearance of Accounting Charges: 7/31/81

Grant/Contract Closeout Actions Remaining:

- [ ] Final Invoice and Closing Documents
- [x] Final Fiscal Report via FCTR
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- [ ] Govt. Property Inventory & Related Certificate
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FORM OCA 10:781

GEORGIA INSTITUTE OF TECHNOLOGY
ATLANTA, GEORGIA 30332

COLLEGE OF
INDUSTRIAL MANAGEMENT

June 29, 1980

M-50-632

Dr. Daniel H. Newlon
Associate Program Director
for Economics Division of Social Studies
National Science Foundation
Washington, D.C. 20550

Dear Dr. Newlon:

I am a little late in sending you an annual progress report of Grant No. Soc - 7820169, Expectations and Equilibrium Over Time. The news is good, however, even if it has been a little slow in developing.

A major concern of the proposed work is the issue regarding the viability of a competitive pricing mechanism in the context of sequential trading with unconditional futures markets, where the possibility of bankruptcy infects the consistency of individual agent choice and the problem of establishing existence of temporary price equilibria. I spent last summer and fall resolving this issue. The price expectations of individual agents and the aggregate of these expectations of course bear the burden of this resolution.

At the end of my previous NSF project, the assumptions that seemed necessary to solve this problem also appeared to have contradictory implications. By the end of summer, 1979, I had finally proved that these assumptions were indeed logically inconsistent. The proof employs a fixed-point argument that produces a result regarding the structure of agent expectations that could not have been anticipated. Fortunately this result also reveals how the original work can be modified to obtain existence of a sequence of temporary price equilibra. These modifications were undertaken during fall, 1979, when the grant paid for a reduced teaching load.

During winter and spring, 1980, I resumed full-time teaching and began writing up the two papers that will contain the economic model and theory developed to date. This is an arduous process and will continue into this summer. I will use this summer and fall (when my teaching load is reduced under terms of this grant) to begin the extensions of the model mentioned in my proposal (section D.3) I am anxious to get at the study of merging of opinions that was part of my original motivation for undertaking this work.

David Welch needs a copy of this type request.
This was filed away without my seeing it
2/16/81

I anticipate, however, that my efforts in pursuing this work will carry beyond 1980 into 1981. Under the arrangements I currently have, I resume full-time teaching in winter 1981. I would greatly appreciate it if the funds in my grant budgeted for graduate students and secreterial - clerical assistance, a sum $12,500, could be used to obtain a reduction in my teaching load for winter and/or spring, 1981. This would involve a transfer of these funds, within personnel services, from B.3 and B.5 on the budget form to A.1 on this form.

Please let me know if this transfer can be made. I am looking forward to hearing from you.

Sincerely yours,

David C. Nachman

David C. Nachman,
Associate Professor

DCN:bw

Milton R. Blood

Milton R. Blood
Associate Dean
College of Management

October 28, 1980

NSF APPROVAL GRANTED

APPROVED:

Leamon R. Scott
Contracting Officer
Office of Contract Administration

Daniel H. Newlon
Economics Program Director

xc: James L. Bostick
Contracting Officer
National Science Foundation

NATIONAL SCIENCE FOUNDATION
Washington, D.C. 20550

FINAL PROJECT REPORT

NSF FORM 98A

PLEASE READ INSTRUCTIONS ON REVERSE BEFORE COMPLETING

PART I–PROJECT IDENTIFICATION INFORMATION

1. Institution and Address	2. NSF Program	3. NSF Award Number
Georgia Institute of Technology Atlanta, Georgia 30332	ECONOMICS	Grant No. SOC-7820169
	4. Award Period From 3/1/79 To 7/31/81	5. Cumulative Award Amount $66802

6. Project Title

EXPECTATIONS AND EQUILIBRIUM OVER TIME

PART II–SUMMARY OF COMPLETED PROJECT *(FOR PUBLIC USE)*

ATTACHED

PART III–TECHNICAL INFORMATION *(FOR PROGRAM MANAGEMENT USES)*

1. ITEM *(Check appropriate blocks)*	NONE	ATTACHED	PREVIOUSLY FURNISHED	TO BE FURNISHED SEPARATELY TO PROGRAM: Check (✓)	TO BE FURNISHED SEPARATELY TO PROGRAM: Approx. Date
a. Abstracts of Theses	X				
b. Publication Citations				X	Indefinite
c. Data on Scientific Collaborators		X			
d. Information on Inventions	X				
e. Technical Description of Project and Results		X			
f. Other *(specify)* None					

2. Principal Investigator/Project Director Name *(Typed)*	3. Principal Investigator/Project Director Signature	4. Date
David C. Nachman	David C. Nachman	10/22/81

NSF Form 98A (5-78) Supersedes All Previous Editions

Form Approved OMB ...

Part II: Summary of Completed Project

The possibility of bankruptcy is a major obstacle to a competitive theory of resource allocation for economies with trading in goods and assets at more than one date. A basic ingredient for such a theory is an economically sensible model of individual choice incorporating a purely competitive notion of solvency and arbitrage restrictions on prices. The demand behavior of individuals derived from this model and the arbitrage restrictions implied by it must then be shown to be consistent in the usual sense that there is a system of prices at each trading date that clears both goods and asset markets, and hence where no agent is bankrupt. The evolution of the economy can then be described as a succession of temporary competitive equilibrium states.

It is clear from previous work that agents' expectations regarding future prices play the dominant role in any theory of temporary competitive equilibrium. The objective of this research is to articulate that role in the context of an exchange economy with spot and limited forward trading at each of a sequence of dates. The major results of this research provide conditions on agents' expectations that ensure the existence of a sequence of temporary equilibria.

First it is shown that, in addition to standard assumptions of temporary competitive analysis, if an agent's expectations regarding future prices are consistent in the sense that he believes that arbitrage on forward markets is impossible, then the agent's choice is determinate and his demand for goods and forward contracts is continuous in the appropriate sense. Second, it is shown that commoness and compatibility assumptions regarding the collection of agents' expectations imply that for important histories at a given date, including histories of temporary equilibria, roughly speaking, every agent believes that any positive spot price system is possible at the succeeding date. This result further implies that for these important histories, individual agent and hence aggregate demand for forward contracts are bounded below by resources at the subsequent date. The two results taken together then lead via standard arguments to the existence of a sequence of temporary equilibria. In addition to these results, some preliminary work has begun on the optimality of such equilibrium paths.

Part III: 1.c. Data on Scientific Collaborators.

Robert P. Kertz

Associate Professor

School of Mathematics

Georgia Institute of Technology

Atlanta, Georgia 30332

Part III. 1.e. Technical Description of Project and Results.

Attached are three papers containing the results of the research funded under Grant No. SOC-7820169. The papers "Consistency and Continuity of Choice in a Sequence of Spot and Futures Markets" and "Temporary Competitive Equilibrium in a Sequence of Spot and Futures Markets" contain the major results of this work. Both papers are under revision for submission for publication. The third paper, "A Note on Valuation Equilibrium and Pareto Optimum", represents some first steps in exploring the optimality of the equilibria established in the paper "Temporary Competitive Equilibrium...."

CONSISTENCY AND CONTINUITY OF CHOICE IN A

SEQUENCE OF SPOT AND FUTURES MARKETS[1]

By

David C. Nachman
and
Robert P. Kertz

CONSISTENCY AND CONTINUITY OF CHOICE IN A SEQUENCE OF SPOT AND FUTURES MARKETS[1]

By David C. Nachman and Robert P. Kertz

This paper is the first part of a competitive analysis of an exchange economy where markets are open at each of an infinite sequence of dates for spot trading and unconditional futures contracting. In the absence of institutional arrangements for handling bankruptcy, the consistency (determinateness) and continuity of agent choice becomes an issue. If an agent's probabilistic opinions (expectations) regarding prices are consistent in an appropriate sense, then choice is consistent and demand is upper hemi-continuous for important price-action histories. In the second part of this analysis [29], commonness and compatibility assumptions regarding agents' opinions imply a specific support structure of these opinions. This structure entails that for important histories at a given date individual and aggregate demand for futures contracts are bounded below by resources at the subsequent date. Existence of a sequence of temporary equilibria then follows in a routine fashion.

1. Introduction

In recent years, considerable attention has been focused on the equilibrium analysis of incomplete market economies [33] with particular emphasis on temporary equilibria in economies that evolve in time [16]. The objective of this research is a competitive analysis of an economy where markets are open at each of an infinite sequence of dates for spot trading and limited unconditional futures trading, and where agents at each date are uncertain only about prices that will prevail at future dates. The problem is the viability of a purely competitive exchange mechanism in this context of sequential trading under uncertainty.

The problem is really twofold, consisting of one problem of an essentially static nature and one of a dynamic nature. The static problem concerns the existence of a temporary equilibrium for markets open at a given date, i.e., at one stage of sequential trading. The usual method of demonstrating existence of a competitive equilibrium is to apply a market equilibrium theorem, e.g., the theorem of Debreu [9]. The application of such a theorem requires that aggregate demand exhibit appropriate behavior over approaches to the boundary of the set of admissible prices (c.f.[16: section 3.1]). This behavior can be deduced if aggregate demand is bounded below independently of prices. Similar boundedness requirements are also needed for application of the more recent results of Mas-Colell [27], Gale and Mas-Colell [14], and Shafer and Sonnenschein [36].

In the usual case, either no uncertainty [10: Chapter 5] or a complete finite system of contingent claims markets [10: Chapter 7], an individual consumer's choice set, and hence demand, are bounded below independently of prices by a priori but natural minimum consumption constraints. In an economy with markets for unconditional forward commitments, individual and aggregate demand for forward contracts need not be bounded below in the appropriate sense unless bounds of an institutional character are imposed. Such bounds are imposed by Stigum [39: (3), p. 541, (2), p. 545], [40: (2.3), (2.8)] in a model with spot markets for currently deliverable goods and for consumer and entrepreneurial debt, and by Radner [32: (2.1), (5.2)] in a model with an incomplete system of contingent claims markets.

In contrast, Green [18] addresses the determinateness of agent choice and boundedness of aggregate demand directly in an exchange model that is the prototype for the one considered here. There are two market dates in this model with spot and unconditional futures markets at the initial date and spot markets at at the second or final date. Green [18: Theorem 2.1] characterizes price systems for the initial date for which the choice behavior of an agent at that date is determinate.

Moreover, Green shows that the desired behavior of aggregate demand in this model can be obtained without institutional bounds provided agent's opinions about prices exhibit some degree of commonness.[2] Commonness of opinions in an ordinary sense of existence of common price forecasts (based on current prices) is a necessary condition for aggregate demand to be well defined, i.e., to have a non-empty domain [18: (4.1)]. Beyond this, Green's condition [18: (4.2)] requires that agents share an open set of price forecasts for the second date that is independent of initial date prices.

The dynamic problem is posed at the end of [18]. The problem is that of determining conditions, if any, under which a history of temporary equilibrium actions and prices up through one stage of sequential trading gives rise to initial conditions at the succeeding stage that ensure existence of a temporary equilibrium at that stage. At each stage of trading, agents must honor contractual obligations incurred at previous stages. Thus the dynamic problem is one of determining conditions under which temporary equilibrium actions at one stage entail contracts that can be honored at subsequent stages. This is the general problem of bankruptcy.

At the date in question in [39], there are preexisting contractual obligations in the form of maturing consumer and entrepreneurial loans. Stigum establishes existence of a temporary equilibrium [39: Proposition III] under conditions that assume away a possible bankruptcy [39: (1), (2), p. 548]. These conditions are relaxed considerably in [40: Propositions 1, 2] to essentially the same effect. There is no guarantee, however, that these conditions would or could obtain as a consequence of a history of temporary equilibria.[3]

In [1: Theorem T.5.7], Arrow and Hahn establish the existence of a compensated equilibrium with bankruptcy for an economy with preexisting personal or household debt. They point out the difficulty in defining a corresponding notion of competitive equilibrium with bankruptcy [1: Reamark 1, p. 121]. Grandmont [15] and Green [19] avoid this issue altogether by investigating equilibrium in competitive type models augmented by institutional arrangements that may be interpreted as bankruptcy laws. Agents take into account the bankruptcy law in deciding to become debtors or creditors [15] or in making unconditional futures contracts for sale or delivery of goods [19]. The outcomes of these augmented mechanisms reflect then not only the action of the bankruptcy law in case of bankruptcies but they also reflect the influence of the bankruptcy law on agents' decisions.

While it is desirable to study alternative institutional arrangements for handling bankruptcies, such investigations may shed little light on the circumstances in which these arrangements may be required. Some understanding of why a competitive mechanism fails to avoid bankruptcies is needed. In the context of sequential trading, the burden of avoiding bankruptcies must be carried by agents' opinions. In the tradition of general competitive analysis, a step toward such understanding is made by answering the question, are there conditions on agents' opinions that imply the existence of a temporary equilibrium path in a sequential trading model with no institutional arrangements for handling bankruptcy? "In attempting to answer the question 'Could it be true?', we learn a good deal about why it might not be true."[4]

This question is addressed here and in a subsequent paper [29] in the context of an extension of Green's model in [18] to an infinite sequence of market dates. The question is broken into two parts. The first, the subject of this paper, concerns the consistency and continuity of individual agent choice at each date and over the infinite horizon. Assumptions made regarding an agent's preferences and endowments are more or less standard in the literature of temporary equilibrium theory. In addition, two assumptions are made regarding an agent's opinion. The first consists of four hypotheses, three of which parallel assumptions in the literature. The fourth is a new (to the best of our knowledge) continuity hypothesis that requires the boundary of the support of an agent's (date n) opinion to have subjective probability zero. The second assumption also has no counterpart in the extant literature, though it is consistent with the interpretation of futures prices as forecasts of future spot prices. This assumption and the new continuity hypothesis imply that at each date and history at that date the set of prices for the subsequent date at which an agent's set of feasible actions is bounded has subjective probability one.

With regard to consistency (c.f. [24]) and even continuity of agent choice, these assumptions are clearly of a sufficiency character and no attempt has been made to investigate their necessity. For the sake of convenience, however, and at the risk of abusing the language, opinions that satisfy these assumptions are referred to simply as consistent. The assumptions are discussed further in Section 2 where they are presented.

In Section 3, the feasible action relations for a typical agent are defined and the regularity properties of these relations are derived.

Included here are some remarks on the concept of economic bankruptcy.

In Sections 4 and 5, the agent's choice problem is cast in the framework of (non-stationary) dynamic programming. The approach, as in [24], builds on the work of Jordan [23]. The results of [23] and [24], however, cannot be applied here directly owing to the possibility of empty values of the feasible action relations and the lack of continuity of these relations at some state (price)-action histories. In Section 4, the relation between histories and attainable futures (probabilities on posterities) is defined and the needed regularity properties of this relation are derived. In Section 5, these results are used to establish existence of solutions of an agent's infinite horizon choice problem and to derive continuity, concavity/convexity, and monotonicity properties of derived utility and single period demand relations.

The second part of the above question, dealt with in [29], concerns the consistency (in the aggregate sense) of individually optimal sequential choice decentralized through the price mechanism at each market date. Two assumptions are made concerning agents' opinions that ensure the existence of appropriate common price forecasts. The first is necessary for aggregate demand to be well defined while the second can be viewed as a dynamic analog of Green's commonness of expectations condition. A third assumption of a compatibility nature is made on opinions to obtain a set of admissable prices suitable for application of a market equilibrium theorem.

These assumptions, together with those of this paper, imply a specific support structure of agents' opinions. As a consequence of this specific structure, at important histories each agent's demand for futures contracts on current markets is bounded below by the agent's known endowment at the subsequent market date. This result solves at once both the static and dynamic problems mentioned earlier. At any candidate for temporary equilibrium at the given date, agents individually choose actions that entail contracts they can honor at the subsequent market date. Aggregate demand for such contracts is bounded below by the resources of the economy available at the next market date, and the existence of a sequence of temporary equilibria for the economy follows in a routine fashion.

We begin with some notation and terminology. Results, definitions, remarks, important conventions, etc., are numbered consecutively within each subsection of the paper.

2. The Model and the Assumptions

2.1. Notation and Terminology

Let $\mathbb{R}$ denote the set of real numbers, $\mathbb{N}$ the set of positive integers, and for m a positive integer, let $\mathbb{R}^m$ denote the m-fold Cartesian product of $\mathbb{R}$ with itself. If $x = (x_1,\ldots,x_m) \in \mathbb{R}^m$ and $y = (y_1,\ldots,y_m) \in \mathbb{R}^m$, $x \leqq y$ (or $y \geqq x$) means $x_i \leq y_i$, $i = 1,\ldots,m$, $x \leq y$ $(y \geq x)$ means $x \leqq y$ and $x \neq y$, and $x < y$ $(y > x)$ means $x_i < y_i$, $i = 1,\ldots,m$. We let $\mathbb{R}^m_+ = \{x\colon x \in \mathbb{R}^m, x \geqq 0\}$, $\mathbb{R}^m_{+0} = \{x\colon x \in \mathbb{R}^m, x \geq 0\}$, and $\mathbb{R}^m_{++} = \{x\colon x \in \mathbb{R}^m, x > 0\}$, where $0 \in \mathbb{R}^m$. These relations and sets are defined in an analogous coordinatewise fashion on $\mathbb{R}^\infty = \mathbb{R} \times \mathbb{R} \times \cdots$. If $x = (x_1,\ldots,x_m) \in \mathbb{R}^m$, $\|x\|$ denotes the Euclidean norm of x, while $\tilde{x} = x(\Sigma_{i=1}^m |x_i|)^{-1}$ provided $x \neq 0$. Similarly, for $B \subseteq \mathbb{R}^m$, $\tilde{B} = \{z\colon z = \tilde{x}, x \in B, x \neq 0\}$.

For a topological space X, $\mathcal{B}(X)$ denotes the σ-algebra of Borel subsets of X, and $\mathcal{P}(X)$ denotes the set of probability measures defined on $\mathcal{B}(X)$, endowed with the topology of weak convergence.[5] Unless specifically stated to the contrary, products of topological spaces are given the product topology and products of measurable spaces are given the product σ-algebra [2: 2.6.1, 2.7.1].

Notation and terminology concerning functions, realtions, and correspondences follows that in [20: pp. 4-5, 21-28], except that we use the notation $g: X \to Y$ to denote the former and $g: X \twoheadrightarrow Y$ to denote the latter, where X and Y are sets. Throughout, the abbreviations "u.h.c." and "l.h.c." stand for "upper hemi-continuous" and "lower hemi-continuous", respectively [20: pp. 21-28].

For two measurable spaces $(\Omega, \mathscr{F})$ and $(\Omega', \mathscr{F}')$, by a transition probability from Ω to Ω' we mean a function $p: \Omega \times \mathscr{F}' \to [0,1]$ such that for each ω in Ω, $p(\omega, \cdot)$ is a probability measure on $\mathscr{F}'$ and for each A in $\mathscr{F}'$, $p(\cdot, A)$ is $\mathscr{F}$-measurable ($\mathscr{F}/\mathbb{B}([0,1])$ - measurable). Alternatively, if $\mathscr{P}(\Omega', \mathscr{F}')$ denotes the set of probability measures on $\mathscr{F}'$, a transition probability from Ω to Ω' is a function $p: \Omega \to \mathscr{P}(\Omega', \mathscr{F}')$ such that for each A in $\mathscr{F}'$, $p(\cdot)(A)$ is $\mathscr{F}$-measurable.

Some basic sets and spaces of interest in this paper are listed here. Their intended interpretations will become clear as the analysis develops. Let ℓ be a positive integer, $\ell \geq 2$ (the number of commodities). For each positive integer n, let $C_n = \mathbb{R}_+^\ell$, $F_n = \mathbb{R}^\ell$, $A_n = C_n \times F_n$, $S_n = \mathbb{R}^{2\ell}$, $H_n = A_{n-1} \times S_n$, $n \geq 2$, and $H_1 = S_1$. The Cartesian product of any of these sets through n is indicated with a subscript "(n)". Thus $S_{(n)} = S_1 \times \cdots \times S_n$ and $H_{(n)} = H_1 \times \cdots \times H_n$. The Cartesian product of any of these sets from n on is indicated with a superscript "(n)". Thus $C^{(n)} = C_{n+1} \times C_{n+2} \times \cdots$ and $H^{(n)} = H_{n+1} \times H_{n+2} \times \cdots$. The symbol for a space without a sub- or superscript denotes the infinite product space, e.g., $C = C_1 \times C_2 \times \cdots$ and $H = H_{(n)} \times H^{(n)}$ for each n. For any product space having $S_n (A_n)(C_n)(F_n)$ as a factor, $\xi_n (\alpha_n)(\alpha_n^1)(\alpha_n^2)$ will denote the projection of the product onto $S_n (A_n)(C_n)(F_n)$. The map $\xi_n^1 (\xi_n^2)$ denotes the projection of any product space with S_n as a factor onto the first (second) ℓ coordinates of S_n.

2.2. Markets, Commodities, and Prices

We consider an exchange economy where markets are open at each of an infinite sequence of dates, indexed by $\mathbb{N}$, for spot trading and limited futures trading. For simplicity, it is assumed that there are $\ell \geq 2$ elementary commodities traded at each date. Elementary commodities are those distinguishable by their physical characteristics and perhaps location but not by date or state of nature. At each date, futures contracts may be executed for unconditional receipt or delivery of each elementary commodity one period hence.[5] Here a period is simply the time elapsed between successive market dates.

The terms of exchange at each date n are expressed by a price system, in units of account, that assigns to each elementary commodity j two real numbers s_{nj} and $s_{n(\ell+j)}$, where s_{nj} is the spot price and $s_{n(\ell+j)}$ is the futures price of commodity j. To say that at date n the price system $s_n = (s_{n1},..,s_{n2\ell})$ prevails means the following. To obtain a unit of commodity j an agent will in general have to trade other commodities or claims to other commodities. To obtain delivery at date n of one unit of commodity j in terms of current (spot) delivery of commodity k, an agent must deliver to the spot market at n s_{nj}/s_{nk} units of commodity k, provided $s_{nk} \neq 0$. If $s_{nk} = 0$, exchange for commodity j in terms of current delivery of commodity k is not possible. To obtain delivery at date n of one unit of commodity j in terms of a future claim on commodity k, an agent must contract to deliver to the spot market at date $n + 1$ $s_{nj}/s_{n(\ell+k)}$ units of commodity k, provided $s_{n(\ell+k)} \neq 0$. If $s_{n(\ell+k)} = 0$, exchange for commodity j in terms of future delivery of commodity k is not possible. The exchange ratios $s_{n(\ell+j)}/s_{nk}$ and $s_{n(\ell+j)}/s_{n(\ell+k)}$ have similar interpretations when they are defined. If $s_{nj} = 0$, exchange in terms of commodity j if $j \leq \ell$ or in terms of future claims on commodity $j - \ell$ if $j > \ell$ is not possible.

It appears from these interpretations that what is important for agents' trade decisions is relative prices. This is in fact the case in standard general equilibrium models [10] primarily because there is no primitive data of these models that depend on prices. In models of temporary general equilibrium, however, agents' opinions are data and these opinions at any date depend, in general, on the past history of prices and on the prevailing price system. The past history and prevailing price system enter an agent's derived utility function through conditional expectation (see Section 5). In this case, the all importance of relative prices comes into question.

From an economic modeling point of view, there are two sides to this question. First, one must determine the set of price histories, including current or prevailing prices, for which each agent must be assumed to respond by the formation of an opinion regarding future prices. Secondly, one must specify the range of these opinions. In models of temporary equilibrium, there is every reason not to restrict consideration on either side to normalized or relative prices. Not only is demand, in general, not positively linear homogeneous in prices, but such restrictions preclude studying price level dynamics of sequences of temporary equilibria and formulating and studying hypotheses of inflationary and deflationary expectations. While these price level issues are not immediate objectives of this work, the model developed here may provide the foundation for such investigations.

The price space for each date n is therefore taken to be S_n. Although in principle agents will be assumed to have a well defined opinion regarding future prices at date n for every history $s_{(n)}$ in $S_{(n)}$, for purposes of analysis, only price systems in $R_{+0}^{2\ell}$ will matter. The role played by normalized prices in the sequel will be strictly a technical one.

2.3. Agent Characteristics

An agent is viewed as an idealized decision making unit that engages in trade at each date, and hence lives forever, and faces uncertainty at each date only regarding prices that will prevail at future market dates. The typical agent is identificed by a 4-tuple $\langle b_0, \omega, \precsim, q \rangle$, where $b_0 = (b_{01},\ldots,b_{0\ell}) \in \mathbb{R}^\ell$, $\omega = \{\omega_n\}_{n\in\mathbb{N}} \in C$, $\precsim \subseteq \mathcal{P}(C) \times \mathcal{P}(C)$, and $q = \{q_n\}_{n\in\mathbb{N}}$, is a sequence of transition probabilities $q_n\colon H_{(n)} \times A_n \to \mathcal{P}(S_{n+1})$, each n in $\mathbb{N}$.

The vector b_0 represents the preexisting contracts of the agent that must be honored at date 1. If $b_{0j} < 0$, the agent has contracted to deliver $-b_{0j}$ units of commodity j at date 1 and if $b_{0j} > 0$, the agent has contracted to receive b_{0j} units of commodity j at date 1.

The sequence ω is a naturally occurring endowment posterity, with ω_n interpreted as a vector of elementary commodities made available to the agent at date n. In general, at any date prior to n, an agent would be uncertain regarding ω_n. For simplicity however, we assume that ω is known with certainty at date 1. A more realistic point of view would be to assume that an agent's endowment sequence is a predictable stochastic process or to assume predictability of some lower bounds for the stochastic process of endowments (c.f. [31]). These generalizations form the basis of research being conducted by the first author on sequences of random temporary equilibria and their asymptotic behavior.

Assumption (A.1): $\omega_1 + b_0 \in \mathbb{R}^\ell_{+0}$ and $\omega_n \in \mathbb{R}^\ell_{++}$, $n \geq 2$.

Remarks (2.3.1): The quantity $\omega_1 + b_0$ is the agent's naturally occurring endowment for date 1 augmented by the vector of preexisting contracts. The assumption that $\omega_1 + b_0 \geq 0$ is an initializing condition. This assumption together with strict positivity of ω_n, $n \geq 2$, is needed in obtaining lower hemi-continuity of feasible action relations (Section 3). Some such positivity or semi-positivity conditions are common in general equilibrium models.

The relation $\precsim$ is a preference ordering that is assumed to satisfy the usual version of the expected utility hypothesis [16: Assumption 1], and a risk aversion and monotonicity hypothesis [18: 2.4(b), (c)].

Assumption (A.2): There exists a bounded continuous function $u:C \to \mathbb{R}$ such that (i) if $\nu^1, \nu^2 \in \mathcal{P}(C)$, $\nu^1 \precsim \nu^2$ if and only if $\int u d\nu^1 \leq \int u d\nu^2$;[7] (ii) u is concave on C; and (iii) u is strictly increasing on C, i.e., if $c^1, c^2 \in C$ and $c^1 \leq c^2$, then $u(c^1) < u(c^2)$.

The sequence $q = \{q_n\}_{n\in\mathbb{N}}$ of transition probabilities is referred to as an opinion and its $n^{\underline{th}}$ component q_n as a date n opinion. For each history $(h_{(n)}, a_n)$, $q_n(h_{(n)}, a_n)$ is the agent's subjective probability measure over date $n + 1$ (equilibrium) price systems. The date n opinion q_n is the analog here of the agent's "expectation function" [16: p. 539].

Definitions (2.3.2): For each integer n, define the correspondence $\sigma_n:H_{(n)} \times A_n \Rightarrow S_{n+1}$ by $\sigma_n(h_{(n)}, a_n) = \text{supp}(q_n(h_{(n)}, a_n))$, where $\text{supp}(\mu)$ denotes the support of the measure μ [30: p. 28]. While σ_n has closed values, these values need not be bounded. For technical purposes, it will suffice to work with a bounded transformation of σ_n. For $j = 1,2$, let $e^j = (1,\ldots,1) \in \mathbb{R}^{j\ell}$ and let $\Omega^j = \{x \in \mathbb{R}^{j\ell}: x \cdot e^j = 1\}$, where as usual $x \cdot y = \Sigma_{k=1}^{j\ell} x_k y_k$. For each n, define the relation $\bar{\sigma}_n:H_{(n)} \times A_n \Rightarrow \Omega^2$ by $\bar{\sigma}_n(h_{(n)}, a_n) = \text{cl}(\tilde{\sigma}_n(h_{(n)}, a_n))$, where "cl" denotes closure.[8] Then $\bar{\sigma}_n$ is compact valued and $\bar{\sigma}_n(h_{(n)}, a_n) = \emptyset$ if and only if $q_n(h_{(n)}, a_n)(\{0\}) = 1$, i.e., $q_n(h_{(n)}, a_n)$ is degenerate at 0 in $\mathbb{R}^{2\ell}$.

Conventions (2.3.3): Henceforth, unless specifically stated otherwise, the quantifier "for each n in $\mathbb{N}$" is understood in all definitions, assumptions, theorems, etc., in the sequel. Also, if the range of a variable is left unspecified in such statements it is understood to be the relevant projection or factor space of H. For example, in (A.3)(i) below, the range of $(h_{(n)}, a_n)$ and $(h'_{(n)}, a'_n)$ is the set $H_{(n)} \times A_n$.

Assumption (A.3): (i) If $\xi_k(h_{(n)}, a_n) = \xi_k(h'_{(n)}, a'_n)$, $k = 1,\ldots,n$, then $q_n(h_{(n)}, a_n) = q_n(h'_{(n)}, a'_{(n)})$; (ii) q_n is continuous and the values of σ_n are convex subsets of $\mathbb{R}^{2\ell}_+$; (iii) $\bar{\sigma}_n$ is u.h.c. on $H_{(n)} \times A_n$; and (iv) $q_n(\text{int}\ \sigma_n(h_{(n)}, a_n)|h_{(n)}, a_n) = 1$.

Remark (2.3.4): Assumption (A.3)(i) states simply that opinions do not depend on the history of past and current actions. This assumption is implicit in most models of temporary competitive equilibrium reflecting the underlying assumption that agents are price takers, who, by definition, believe that their actions have no effect on price formation. We will alternately write q_n and objects derived from it, such as σ_n and $\bar{\sigma}_n$, as functions of the entire history or as functions defined on $S_{(n)}$ as convenience dictates.

Remarks (2.3.5): The continuity assumption of (A.3)(ii) is also standard in temporary equilibrium models [16: Assumption 2], [18: (3.1)]. As in [18: Remark 3.1], this continuity assumption implies that σ_n is l.h.c. It then follows from (A.3)(ii)-(iv) that $\bar{\sigma}_n$ is non-empty ((iii) says this implicitly, but so does (iv)), compact, and convex valued and continuous on $H_{(n)} \times A_n$, since the mapping $x \to \tilde{x}$ is continuous on $\mathbb{R}^{2\ell}_{+0}$ and preserves convexity.

Remarks (2.3.6): From a strictly mathematical point of view, the convexity assumption of (A.3)(ii) is probably unnecessary since most of the results of this paper would hold without this assumption if everywhere σ_n is replaced by co σ_n, the convex hull of σ_n. This greater generality, however, leads to the possibility (in [29]) of a temporary equilibrium price system, say at date $n \geq 2$ for a given history, for which for some agent this price system belongs to co σ_{n-1} but not to σ_{n-1} at that history. Such a possibility seems inconsistent with the interpretation of agent opinions as probabilistic forecasts of future price equilibria and precludes the possibility that opinions are even in the weakest sense realized or fulfilled at equilibrium. The convexity assumption simply avoids this issue and simplifies the analysis as well. The non-negativity assumption of (A.3)(ii) is consistent with (A.2)(iii) that implies that all commodities are desired [18:(2.3)(ii)]

Remarks (2.3.7): The convexity assumption of (A.3)(ii) also rules out date n opinions with finite non-degenerate support, as does (A.3)(iv), which implies that int σ_n is non-empty valued. (A.3)(iv) is a continuity hypothesis similar in spirit to those made about consumer preferences [10: Section 4.6] It is used in establishing lower hemi-continuity of the relation studied in Section 4. Opinions absolutely continuous with respect to Lebesgue measure or opinions with atoms that form a countably dense subset of some open convex set are consistent with (A.3)(iv). Assumption (A.3)(iii) is also a continuity hypothesis analogous to [18: (3.2)].

Remark (2.3.8): Let $\sum_n^o$ denote the graph of int σ_n. Under (A.3)(ii),(iv), $\sum_n^o$ is open in $H_{(n+1)}$. For by (2.3.5), int σ_n is a l.h.c. correspondence. It then follows that the triple $(H_{(n)} \times A_n, S_{n+1}$, int $\sigma_n)$ satisfies [8: CST, p.772] and by the remark following [8: Theorem 2], $\sum_n^o$ is open in $H_{(n)} \times A_n \times S_{n+1} = H_{(n+1)}$.

Definitions (2.3.9): In light of (A.3)(iv), price systems s_{n+1} in int $\sigma_n(s_{(n)})$ are referred to as forecasted price systems, price forecasts, or simply as forecasts (at the price history $s_{(n)}$). If $x \in \mathbb{R}^\ell$, x is a spot (futures) price forecast if $x = \xi^1_{n+1}(s_{n+1})(\xi^2_{n+1}(s_{n+1}))$ for some price forecast s_{n+1}. Similarly, if $x = \tilde{s}_{n+1}$ for a price forecast s_{n+1}, then x is referred to as a relative price forecast. For such x, $\gamma\xi^1_{n+1}(x)$ is a spot price forecast and $\gamma\xi^2_{n+1}(x)$ is a futures price forecast for some scalar γ.

Definitions (2.3.10): Of particular interest in analyzing agent choice is the set of current price systems whose futures price component is a spot price forecast for the subsequent date, appropriately scaled. For $B \subseteq \mathbb{R}^m$, $B \neq \emptyset$, let $\Gamma(B)$ denote the smallest cone containing B. For n = 1, define

$$Q_1 = \{s_1 : \xi^1_1(s_1) \in \mathbb{R}^\ell_{++}, \xi^2_1(s_1) \in \xi^1_2(\Gamma(\text{int } \sigma_1(s_1)))\} ,$$

and for $n \geq 2$, let $Q_n : S_{(n-1)} \Rightarrow S_n$ be defined by

$$Q_n(s_{(n-1)}) = \{s_n : \xi^1_n(s_n) \in \mathbb{R}^\ell_{++}, \ \xi^2_n(s_n) \in \xi^1_{n+1}(\Gamma(\text{int } \sigma_n(s_{(n-1)}, s_n)))\} .$$

Then Q_n associates to a price history $s_{(n-1)}$ the set of date n price systems s_n that are positive and whose futures price component $\xi^2_n(s_n)$, perhaps after some scaling, is a spot price forecast for date n+1 at the history $(s_{(n-1)}, s_n)$. A similar interpretation applies to the set Q_1, which is the analog here of the set S in [18: p. 116].

Remark (2.3.11): It follows from (2.3.8) by a straight forward argument that Q_1 is an open subset of S_1 and that for $n \geq 2$, Q_n is open valued and has an open graph in $S_{(n)}$.

The importance of the price systems in $Q_n(s_{(n-1)})$ is that at these prices and only at these prices is an agent's set of feasible actions at date n bounded (Theorem (3.2.2)). In light of this result, it seems reasonable to require that $Q_1 \neq \emptyset$ and that for $n \geq 2$ and each $s_{(n-1)}$, $q_{n-1}(Q_n(s_{(n-1)})|s_{(n-1)}) = 1$. If this latter condition holds, then by (A.3)(iv), $\text{int}\ \sigma_{n-1}(s_{(n-1)}) \subseteq \text{int}\ (\text{cl}\ Q_n(s_{(n-1)}))$. For technical reasons it is convenient to have $\text{int}\ \sigma_{n-1}(s_{(n-1)}) \subseteq Q_n(s_{(n-1)})$. This would be the case under the assumption that $Q_n(s_{(n-1)})$ has subjective probability one if this set is a regular open set, e.g., if it is convex. These conditions, however, are not necessary or interpretable, and we assume the desired result.

<u>Assumption</u> (A.4). $Q_1 \neq \emptyset$ and for $n \geq 2$ and $s_{(n-1)}$ in $S_{(n-1)}$, $\text{int}\ \sigma_{n-1}(s_{(n-1)}) \subseteq Q_n(s_{(n-1)})$.

<u>Remark</u> (2.3.12): In the presence of (A.3)(i), the non-negativity and convexity parts of (A.3)(ii), and (A.3)(iv), the inclusion assumed in (A.4) is implied by a version of the expectations hypothesis that, for positive current futures prices, conditional expected spot prices equal current futures prices. To formulate this hypothesis, let $q = \{q_n\}_{n \in \mathbb{N}}$ be an opinion satisfying the parts of (A.3) just mentioned, and let $q_o \in \mathcal{P}(S_1)$ be some (subjective) distribution for initial date prices. The sequence $\{q_n\}_{n \geq 0}$ determines a unique probability measure on $(S, \mathbb{B}(S))$ [2: Theorem (2.7.2)], call it P, and for each $n \geq 1$, q_n is a regular conditional distribution for P of ξ_{n+1} given $\xi_1, \ldots, \xi_n$ [5: p. 79], where these functions are the projections of the infinite product space S onto the factors S_{n+1} and $S_1, \ldots, S_n$, respectively. Thus a version of the (vector of) conditional expectation(s) under P of ξ_{n+1} given $\xi_1, \ldots, \xi_n$ is given by the composition of the vector integral $\int s_{n+1} q_n(ds_{n+1}|s_1, \ldots, s_n)$ (which exists by non-negativity of the support $\sigma_n(s_1, \ldots, s_n)$) with the mapping $s \to (\xi_1(s), \ldots, \xi_n(s))$. Suggestively, let

$E(\xi_{n+1}|\xi_1,\ldots,\xi_n)$ denote this version. Clearly, the projection of $E(\xi_{n+1}|\xi_1,\ldots,\xi_n)$ onto its first ℓ coordinates is a version of the conditional expectation of date $n+1$ spot prices, ξ^1_{n+1}, given $\xi_1,\ldots,\xi_n$, which we also denote suggestively by $E(\xi^1_{n+1}|\xi_1,\ldots,\xi_n)$. The expectations hypothesis is that $E(\xi^1_{n+1}|\xi_1,\ldots,\xi_n)(s) = \xi^2_n(s)$ for every $s \in S$ such that $\xi^2_n(s) > 0$ (note the integrability assumption implicit here). (More generally, one could have $E(\xi^1_{n+1}|\xi_1,\ldots,\xi_n) = \lambda\xi^2_n$ for some positive scalar function λ defined on S and measurable with respect to $\xi_1,\ldots,\xi_n$.) But under the convexity assumption of (A.3)(ii) and (A.3)(iv), it follows that $E(\xi_{n+1}|\xi_1,\ldots,\xi_n)(s) \in \text{int}\ \sigma_n(\xi_1(s),\ldots,\xi_n(s))$ for every $s \in S$. Thus from the expectations hypothesis if $\xi^2_n(s) > 0$, then $\xi^2_n(s) \in \xi^1_{n+1}(\text{int}\ \sigma_n(\xi_1(s),\ldots,\xi_n(s)))$ (here using ξ^1_{n+1} as the projection of S_{n+1} onto its first ℓ coordinates). In particular, if $n \geq 2$ and $\xi_n(s) \in \text{int}\ \sigma_{n-1}(\xi_1(s),\ldots,\xi_{n-1}(s))$, then $\xi_n(s) \in Q_n(\xi_1(s),\ldots,\xi_{n-1}(s))$, as assumed in (A.4).

Remark (2.3.13): The expectations hypothesis just formulated violates the continuity assumption of (A.3)(ii) if the parts of (A.3) assumed in (2.3.12) are maintained. For suppose that $s^k = (s_1,s_2,\ldots) \in S$, $k = 0,1,\ldots,$ with s^k converging to s^0, $s^k_n = \xi_n(s^k) > 0$, $k = 1,2,\ldots,$ and $\xi^2_{nj}(s^0) = 0$, some $j = 1,\ldots,\ell$, where ξ^2_{nj} is the $j^{\underline{th}}$ coordinate of ξ^2_n. Let $q^k_n = q_n(s^k_1,\ldots,s^k_n)$, $k = 0,1,\ldots,$ and let ξ^1_{n+1j} denote the $j^{\underline{th}}$ coordinate of ξ^1_{n+1}. The expectations hypothesis entails that $\lim_k \int \xi^1_{n+j} dq^k_n = \lim_k \xi^2_{nj}(s^k) = 0$, and hence, by the non-negativity in (A.3)(ii), that the sequence of functions $f_k \equiv \xi^1_{n+1j}$ on the measure spaces $(S_{n+1}, \mathbb{B}(S_{n+1}), q^k_n)$, $k = 1,2,\ldots,$ is uniformly integrable [20: (40), p. 52]. But under the continuity assumption of (A.3)(ii), q^k_n converges weakly to q^0_n. It follows that $\int \xi^1_{n+1j} dq^0_n = \lim_k \int \xi^1_{n+1j} dq^k_n = 0$ [20: (42), p. 52]. But this is impossible under the rest of (A.3)(ii) unless the marginal distribution of ξ^1_{n+1j} under q^0_n is degenerate at zero, which violates (A.3)(iv).

The dilemma posed by these last remarks is a dramatic example of an ex ante version of a fundamental problem raised by Jordan [22: pp. 455-456]. In the present case this problem amounts to the choice between positivity of subjective opinions about equilibrium prices and the continuity of these opinions on the boundary of positive prices on the one hand and the ex ante unbiasedness of current futures prices as forecasts of future spot prices on the other hand. If one insists on the former, one cannot have the latter, i.e., expectations about (the unbiased estimates of) future spot prices must differ from current futures prices at least in some neighborhood of each boundary point of positive current futures prices.

While the expectations hypothesis has considerable intuitive appeal and it does explicate the inclusion assumed in (A.4), it is by no means compelling either as an a priori hypothesis, the relevant case here, or as a condition of equilibrium in a competitive setting with or without rational expectations.[9] One might avoid ruling out the expectations hypothesis in the context here by foregoing continuity of q_n on the boundary of $\mathbb{R}^{2\ell}_{++}$ while maintaining lower hemi-continuity of σ_n on this boundary, but we have not explored this possibility. Alternatively, one might modify the hypothesis so that it holds, for date n, only for those price systems in int $\sigma_{n-1}(\cdot)$, for this is all that is required to produce (A.4). In anticipation of [29: Theorem (3.2.2)], however, the contradiction produced in Remark (2.3.13) would remain.

Finally, we note that (A.4) is itself a weak form of a restricted expectations hypothesis. It states that the futures component of every price forecast for date n is, up to scale change, a forecast of spot prices for date n + 1, where the term "forecast" is used in the weak sense of (2.3.9) as opposed to conditional mean forecasts. This provides sufficient intuition for this work to proceed on the basis of (A.3) and (A.4). That these assumptions are consistent can be seen by a trivial example. Let $q_n \equiv \nu \in \mathcal{P}(\mathbb{R}^{2\ell})$ with $\nu(\mathbb{R}^{2\ell}_{++}) = 1$ and $\text{supp}(\nu) = \mathbb{R}^{2\ell}_{+}$. Then (A.3) holds, $Q_1 \equiv Q_n(s_{(n-1)}) \equiv \mathbb{R}^{2\ell}_{++}$, and consequently (A.4) holds as well.

3. Feasible Actions

3.1. Bankruptcy and Feasibility

At each date n, when feasible, an agent chooses a vector c_n in C_n of elementary commodities for consumption and a vector $b_n = (b_{n1},\ldots,b_{n\ell})$ of futures contracts, where $b_{nk} > (<) \; 0$ indicates a contract to receive (deliver) $|b_{nk}|$ units of commodity k at date n + 1. To determine feasibility, agents use the prevailing price system at date n to present value their known endowment augmented by previous contractual obligations.

Conventions (3.1.1): For each n, let $f_{n-1} = \omega_n + b_{n-1}$. Then f_{n-1} is a vector of net futures contracts made at date n-1 corresponding to the vector of futures contracts b_{n-1}, and (f_{n-1}, ω_{n+1}) is the part of the agent's known endowment, augmented by preexisting contracts, relevant for trade at date n. The sense in which f_n represents net futures contracts made at date n is explained below. Since f_n is determined when b_n is and vice versa, we can focus on f_n as the futures part of the agent's action at date n. Formally, an action for an agent at date n is a vector $a_n = (c_n, f_n)$ in A_n, where c_n denotes consumption and f_n denotes net futures contracts. A history $h_{(n)}$ in $H_{(n)}$ for the agent is then a sequence of prices and actions up through prices at date n. For such a history $h_{(n)}$, $\alpha^2{}_{n-1}(h_{(n)})$ is the vector of net futures contracts made at date n-1 with the convention that $\alpha_0^2(h_{(1)}) \equiv \omega_1 + b_0 \equiv f_0$.

Definitions (3.1.2). Define $r_n : H_{(n)} \to \mathbb{R}$ by $r_n(h_{(n)}) = \xi_n(h_{(n)})\cdot(\alpha^2_{n-1}(h_{(n)}), \omega_{n+1})$. Then r_n is agent specific and $r_n(h_{(n)})$ is the present value at prices $\xi_n(h_{(n)})$ of the agent's known augmented endowment at the history $h_{(n)}$. If $r_n(h_{(n)}) \geq 0$, the agent is solvent at $h_{(n)}$, and in this case the agent may execute spot trades and futures contracts subject to the usual budgetary restriction, i.e., the agent may choose an action from the budget set $\hat{D}^1_n(h_{(n)}) \equiv \{a_n \in A_n: \xi_n(h_{(n)}) \cdot a_n \leq r_n(h_{(n)})\}$. If $r_n(h_{(n)}) < 0$, the agent is (defined to be) <u>bankrupt</u> at $h_{(n)}$, and in order to avoid institutional arrangements for handling bankruptcy, the budgetary relation is defined to be empty at $h_{(n)}$. Formally, the budgetary relation $D^1_n: H_{(n)} \Rightarrow A_n$ is defined by

$$\begin{aligned} D^1_n(h_{(n)}) &= \hat{D}^1_n(h_{(n)}), \text{ if } r_n(h_{(n)}) \geq 0, \\ &= \emptyset \ , \text{ if } r_n(h_{(n)}) < 0. \end{aligned}$$

<u>Remark</u> (3.1.3): Clearly r_n is a continuous function and $\text{dom } D^1_n = \{h_{(n)}: r_n(h_{(n)}) \geq 0\}$ is a closed set as is any section of this set.

If there is to be any hope for determinateness of intertemporal choice with the specifications of (3.1.2), an agent must plan to be solvent with subjective certainty.

<u>Definition</u> (3.1.4): Define the relation $D^2_n: H_{(n)} \Rightarrow A_n$ by

$$D^2_n(h_{(n)}) = \{a_n \in A_n: q_n(\text{dom } D^1_{n+1}(h_{(n)}, a_n) | h_{(n)}, a_n) = 1\}.$$

Then $D^2_n(h_{(n)})$ is the set of date n actions where with subjective probability one the agent will be solvent at date n + 1.

<u>Remark</u> (3.1.5): $D^2_n(h_{(n)}) = \{a_n: r_{n+1}(h_{(n)}, a_n, s_{n+1}) \geq 0, \forall s_{n+1} \in \sigma_n(h_{(n)}, a_n)\}$. This follows from the fact that the section $\text{dom } D^1_{n+1}(h_{(n)}, a_n)$ is closed and $\sigma_n(h_{(n)}, a_n)$ is the smallest closed set with $q_n(h_{(n)}, a_n)$ - measure one [30: Theorem 2.1, p.27].

<u>Definition</u> (3.1.6): Feasible actions for the agent at date n are those that satisfy the budgetary relation D_n^1 and the planning relation D_n^2, i.e., the <u>feasible action relation</u> $D_n: H_{(n)} \Rightarrow A_n$ is defined by

$$D_n(h_{(n)}) = D_n^1(h_{(n)}) \cap D_n^2(h_{(n)}).$$

We let Δ_n = graph $D_n = \{(h_{(n)},a_n): a_n \in D_n(h_{(n)})\}$.

Bankruptcy, as we have defined it, is synonomous with negative net worth, and is, as Stigum [41:c10.1 1.2] has noted, somewhat arbitrary from an economic theory perspective. According to this point of view an agent should be declared bankrupt only when he cannot meet his maturing contractual obligations by trading on current markets. The appropriate formulation of the budgetary relation in this case would be simply the budget set relation $\hat{D}_n^1$. An agent would be bankrupt at $h_{(n)}$ in this formulation if and only if $\hat{D}_n^1(h_{(n)}) = \emptyset$, irrespective of the sign of $r_n(h_{(n)})$. For intertemporal consistency then, one would define $\hat{D}_n^2(h_{(n)})$ as in (3.1.4) with dom $\hat{D}_{n+1}^1$ replacing dom D_{n+1}^1.

The problem with this formulation is that under (A.3)(i), if $\Gamma(\Omega^1 \times \{0\}) = \{s_{n+1}: \xi_{n+1}^1(s_{n+1}) \geq 0 = \xi_{n+1}^2(s_{n+1})\}$ has $q_n(h_{(n)},a_n)$- measure zero, which holds under (A.3)(ii),(iv), then $\hat{D}_n^2(h_{(n)}) = A_n$. Thus for $\hat{D}_n^2$ to impose some constraint on choice at date n and history $h_{(n)}$, it is necessary that $\Gamma(\Omega^1 \times \{0\})$ have positive $q_n(h_{(n)},a_n)$- measure, for some and hence all a_n. Establishing existence of temporary competitive equilibrium in this case by the methods employed here and in [29] appears difficult. By these methods, the set $\Gamma(\Omega^1 \times \{0\})$ must, in a sense spelled out in [29], constitute part of the boundary of $\Gamma(\sigma_n(h_{(n)},a_n))$ for appropriate $h_{(n)}$.

The need for $\hat{D}_n^2$ to impose a constraint on date n actions stems from the fact that if $\hat{D}_n^1(h_{(n)}) \neq \emptyset$, this set is not bounded and is unbounded in the direction of increasing preferences (increasing c_n) unless $\xi_n^2(h_{(n)}) = 0 < \xi_n^1(h_{(n)})$. This follows also for D_n^1, but as will be shown below, D_n^2 imposes the needed bounds for appropriate histories. In this regard, the definition of bankruptcy in (3.1.2) seems less arbitrary. We leave for further study the formulation based on $\hat{D}_n^1$.

As a final comment before turning to the analysis of D_n, note that the accounting inequality $\xi_n(h_{(n)})\cdot a_n \leq r_n(h_{(n)})$, in one sense, entails that the agent sell his entire date n + 1 endowment ω_{n+1} on the futures markets at date n. For if $f_{n-1} = \alpha_{n-1}^2(h_{(n)})$ and $a_n = (c_n, f_n)$, then the inequality can be written as $\xi_n^1(h_{(n)})\cdot c_n + \xi_n^2(h_{(n)})\cdot b_n \leq \xi_n^1(h_{(n)})\cdot(\omega_n + b_{n-1})$, where of course $b_{n-1} = f_{n-1} - \omega_{n+1}$. This is the sense in which f_n represents net futures contracts, total futures contracts b_n net of the endowment $-\omega_{n+1}$. Since we impose no restrictions on the choice of f_n that would not be imposed on the quantity $\omega_{n+1} + b_n$ and vice versa, i.e., $f_n \equiv \omega_{n+1} + b_n$ is a true identity throughout our analysis, whether or not the agent must sell his future endowment is a matter of interpretation.

3.2. Regularity of Feasible Actions

In this section we are concerned with regularity of the relation D_n defined in (3.1.6). Throughout this section and the sequel, Assumptions (A.1) - (A.4) are maintained. The first result concerns closedness and convexity of D_n.

Lemma (3.2.1): Δ_n is a non-empty closed subset of $H_{(n)} \times A_n$ and D_n is closed and convex valued on $H_{(n)}$.

Proof: Clearly $0 \in D_n(h_{(n)})$ for every $h_{(n)}$ in dom D_n. By (A.1), $\mathbb{R}_+^{2\ell} \subseteq \text{dom } D_1$, and for $n \geq 2$, $\{h_{(n)}: \alpha_{n-1}^2(h_{(n)}) \in \mathbb{R}_+^{\ell},\ \xi_n(h_{(n)}) \in \mathbb{R}_+^{2\ell}\} \subseteq \text{dom } D_n$. Thus $\Delta_n \neq \emptyset$. The remainder of the lemma follows essentially as in [18: Lemma 3.3] using continuity of ξ_n and r_n and lower hemi-continuity of σ_n. □

The interest in Q_n stems from the following characterization of histories $h_{(n)}$ for which $D_n(h_{(n)})$ is bounded.

Theorem (3.2.2): If $h_{(n)} \in \text{dom } D_n$, then $D_n(h_{(n)})$ is bounded if and only if $\xi_n(h_{(n)}) \in Q_n(\xi_1(h_{(n)}),\ldots,\xi_{n-1}(h_{(n)}))$ ($= Q_1$, for $n = 1$).

Proof: Suppose $h_{(n)} \in \text{dom } D_n$ and that $\xi_n(h_{(n)}) \in Q_n(\xi_1(h_{(n)}),\ldots,\xi_{n-1}(h_{(n)}))$. The proof that $D_n(h_{(n)})$ is bounded is by contradiction, essentially as in [18: Lemma 2.2]. If $D_n(h_{(n)})$ is not bounded, there exists a sequence $\{a^k\}_{k\in\mathbb{N}} \subseteq D_n(h_{(n)})$ such that $\|a^k\| \to \infty$. It follows that either $\|\alpha_n^1(a^k)\| \to \infty$ or $\|\alpha_n^2(a^k)\| \to \infty$. Since $a^k \in D_n(h_{(n)})$ for every k, the valuation inequalities $\xi_n^2(h_{(n)})\cdot\alpha_n^2(a^k) \leq r_n(h_{(n)})$ and $r_{n+1}(h_{(n)}, a^k, (\gamma\xi_n^2(h_{(n)}), p)) \geq 0$ hold for some scalar $\gamma > 0$ and some p in $\mathbb{R}^\ell$ and all k in $\mathbb{N}$. If $\|\alpha_n^2(a^k)\| \to \infty$, then using these inequalities one can find a subsequence $\{a^{k'}\}$ of $\{a^k\}$ and a price system s_{n+1} in int $\sigma_n(h_{(n)}, a^k)$, all k in $\mathbb{N}$, such that $\xi_{n+1}^1(s_{n+1})\cdot\alpha_n^2(a^{k'}) \searrow -\infty$ as $k' \to \infty$. But then $r_{n+1}(h_{(n)}, a^{k'}, s_{n+1}) < 0$ for k' sufficiently large, in contradiction of the fact that $a^k \in D_n^2(h_{(n)})$ for all k.

Thus it must be the case that $\|\alpha_n^1(a^k)\| \to \infty$. But $a^k \in D_n^1(h_{(n)})$ for every k implies that $0 \le \xi_n^1(h_{(n)})\cdot\alpha_n^1(a^k) \le r_n(h_{(n)}) - \xi_n^2(h_{(n)})\cdot\alpha_n^2(a^k) \le M$ for some $M < \infty$. By definition, $\xi_n^1(h_{(n)}) \in \mathbb{R}_{++}^\ell$ and $\alpha_n^1(h_{(n)}) \in \mathbb{R}_+^\ell$, and the contradiction ensues.

To prove the converse, let $h_{(n)}$ be in dom D_n but assume that $\xi_n(h_{(n)}) \notin Q_n(\xi_1(h_{(n)}),\ldots,\xi_{n-1}(h_{(n)}))$. Clearly, if $\xi_n^1(h_{(n)}) \not> 0$, then $D_n(h_{(n)})$ is not bounded. Therefore, assume that $\xi_n^1(h_{(n)}) > 0$. The proof in this case follows [18: Lemma 2.5]. Since $h_{(n)}$ is fixed, let $\Gamma = \Gamma(\sigma_n(\xi_1(h_{(n)}),\ldots,\xi_n(h_{(n)})))$. Then Γ is a convex cone and int $\Gamma = \Gamma(\text{int}\ \sigma_n(\xi_1(h_{(n)}),\ldots,\xi_n(h_{(n)}))) \ne \emptyset$. By assumption, for every $\gamma > 0$ and p in $\mathbb{R}^\ell$, $(\gamma\xi_n^2(h_{(n)}), p) \notin$ int Γ. Let $\Gamma' = \{x \in \mathbb{R}^\ell : (x,y) \in \Gamma$, some $y \in \mathbb{R}^\ell\}$. By [35: Theorem 6.8] and openness of the projection map of $\mathbb{R}^{2\ell}$ onto $\mathbb{R}^\ell$, it follows that for every $\gamma > 0$, $\gamma\xi_n^2(h_{(n)}) \notin \text{ri}\Gamma' = \text{int}\ \Gamma' \ne \emptyset$.

By the usual separation arguments [35: Theorem 11.6], there exists a vector y in $\mathbb{R}^\ell$ such that $y \ne 0$, $\xi_n^2(h_{(n)})\cdot y = 0$, and $x\cdot y \ge 0$ for all $x \in \Gamma'$. It is then easy to show from these properties that $(0,\lambda y) \in D_n(h_{(n)})$ for all λ in $\mathbb{R}$. Thus $D_n(h_{(n)})$ is unbounded.□

Let $Q_1^* = Q_1$ and for $n \ge 2$, let Q_n^* denote the graph of Q_n considered as a subset of $H_{(n)}$, i.e., $Q_n^* = \{h_{(n)} \in H_{(n)} : \xi_n(h_{(n)}) \in Q_n(\xi_1(h_{(n)}),\ldots,\xi_{n-1}(h_{(n)}))\}$.

Theorem (3.2.3): D_n is u.h.c. and compact valued at each point of dom $D_n \cap Q_n^*$.

Proof: By Lemma (3.2.1), (i) Δ_n is closed in $H_{(n)} \times A_n$, (ii) D_n is convex valued on dom D_n, and (iii) $0 \in D_n(h_{(n)})$ for each $h_{(n)}$ in dom D_n. By Lemma (A.1) in the Appendix, D_n is u.h.c. at each $h_{(n)}$ in dom D_n where $D_n(h_{(n)})$ is compact. The result then follows from Theorem (3.2.2).□

Definition (3.2.4). For each n, let $\bar{r}_{n+1}: H_{(n)} \times A_n \to \mathbb{R}$ be defined by $\bar{r}_{n+1}(h_{(n)},a_n) = \min \{r_{n+1}(h_{(n)},a_n,x): x \in \bar{\sigma}_n(h_{(n)},a_n)\}$. Then $\bar{r}_{n+1}(h_{(n)},a_n)$ is the minimum present value, in terms of relative price forecasts and their limits, of the agent's wealth at date n + 1 given the history $h_{(n)}$ and action a_n.

Lemma (3.2.5). $\bar{r}_{n+1}$ is continuous and for each $h_{(n)}$, $\bar{r}_{n+1}(h_{(n)},\cdot)$ is concave on A_n and $a_n \in D_n^2(h_{(n)})$ if and only if $r_{n+1}(h_{(n)},a_n) \geq 0$.

Proof: Follows directly from Remarks (2.3.5) and (3.1.3) and the usual maximum theorems, e.g., [20: Corollary, p.30].□

In establishing where D_n is l.h.c., one encounters a problem analogous to the minimum wealth problem of standard general equilibrium analysis [10: Section 4.8]. As one might expect from this analogy, D_n is l.h.c. wherever r_n is positive. Let $D_n^+ = \{h_{(n)}: r_n(h_{(n)}) > 0\}$.

Theorem (3.2.6). D_n is l.h.c. at each point of D_n^+.

Proof: Let $h_{(n)}$ be in D_n^+ and let $\{h_{(n)}^k\}_{k\in \mathbb{N}}$ be such that $\lim_k h_{(n)}^k = h_{(n)}$.

Since D_n^+ is open, we may assume that $h_{(n)}^k \in D_n^+$ for each k. If $a_n \in D_n(h_{(n)})$, choose f in $\mathbb{R}_{++}^{\ell}$ such that $\xi_n^2(h_{(n)})\cdot f < r_n(h_{(n)})$, and let $a = (0,f)$. Then $a \in D_n^1(h_{(n)})$. Also $(f,\omega_{n+2}) > 0$ and hence $\bar{r}_{n+1}(h_{(n)},a) > 0$. By Lemma (3.2.5), $a \in D_n(h_{(n)})$. For each scalar λ with $0 \le \lambda \le 1$, let $a(\lambda) = \lambda a_n + (1-\lambda)a$. By Lemma (3.2.1), $a(\lambda) \in D_n(h_{(n)})$. In particular, for $0 \le \lambda < 1$, $\xi_n(h_{(n)})\cdot a(\lambda) < r_n(h_{(n)})$ and by Lemma (3.2.5) again, $\bar{r}_{n+1}(h_{(n)},a(\lambda)) > 0$. It is then straight forward, as in [18: Lemma 3.3], to construct a subsequence $\{h_{(n)}^{k'}\}$ of $\{h_{(n)}^k\}$ and a corresponding sequence $\{a^{k'}\}$ with $a^{k'}$ in $D_n(h_{(n)}^{k'})$ such that $\lim_{k'} a^{k'} = a_n$. By [20: Theorem 2, p.27], this completes the proof.□

Remarks (3.2.7). Certainly $\mathbb{R}_+^{\ell} \times \mathbb{R}_{+0}^{\ell} \subseteq D_1^+$, and for $n \ge 2$, $\{h_{(n)}: \alpha_{n-1}^2(h_{(n)}) \in \mathbb{R}_+^{\ell}, \xi_n(h_{(n)}) \in \mathbb{R}_+^{\ell} \times \mathbb{R}_{+0}^{\ell}\} \subseteq D_n^+$. The important histories with regard to regularity of D_n can then be identified as follows. Recall the definition of $\sum_n^o$ in (2.3.8). Define $H_1^* = Q_1^*$, and for $n \ge 2$, define $H_n^* = (\Delta_{n-1} \times S_n) \cap \sum_{n-1}^o$. By Lemma (3.2.1), $H_n^* \ne \emptyset$ for each n.

Lemma (3.2.8). $H_n^* \subseteq D_n^+ \cap Q_n^*$.

Proof: By (A.4) it suffices to show that $H_n^* \subseteq D_n^+$. For $n = 1$ and $s_1 \in Q_1$, $s_1 > 0$ and hence $Q_1 \subseteq D_1^+$. For $n \ge 2$, let $h_{(n)} = (h_{(n-1)},a_{n-1},s_n)$ be in $H_{(n)}^*$. Then $a_{n-1} \in D_{n-1}(h_{(n-1)})$ and hence $r_n(h_{(n)},a_{n-1},s_n') \ge 0$ for all s_n' in $\sigma_{n-1}(h_{(n-1)},a_{n-1})$. If $s_n' \in \operatorname{int} \sigma_{n-1}(h_{(n-1)},a_{n-1})$, then $s_n' > 0$, and if $r_n(h_{(n-1)},a_{n-1},s_n') = 0$, then $\xi_n^1(s_n')\cdot\alpha_{n-1}^2(a_{n-1}) = -\xi_n^2(s_n')\cdot\omega_{n+1} < 0$. By altering an appropriate coordinate of $\xi_n^1(s_n')$ one can produce s_n'' in in $\operatorname{int} \sigma_{n-1}(h_{(n-1)},a_{n-1})$ such that $\xi_n^2(s_n'') = \xi_n^2(s_n')$ and $\xi_n^1(s_n'')\cdot\alpha_{n-1}^2(a_{n-1}) < \xi_n^1(s_n')\cdot\alpha_{n-1}^2(a_{n-1})$. But then $r_n(h_{(n-1)},a_{n-1},s_n'') < 0$, a contradiction. It follows that $r_n(h_{(n-1)},a_{n-1},s_n) > 0$, and the proof is complete. □

4. Attainable Futures

4.1. The Dynamic Programming Problem

In general, an agent's choice of an action at any market date n can depend on the entire history $h_{(n)}$ experienced by the agent up to date n. As is common in formulations of sequential decision processes, it is convenient to allow the agent to make this choice by randomizing over feasible actions.

Definitions (4.1.1). A date n strategy is any function $\pi_n: H_{(n)} \to \mathcal{P}(A_n)$ such that π_n is a transition probability from $H_{(n)}$ to A_n. In the context here (see Lemma (A.II)), such a function π_n is a transition probability if and only if π_n is $\mathbb{B}(H_{(n)})/\mathbb{B}(\mathcal{P}(A_n))$ - measurable. A date n strategy π_n is said to be feasible if $\pi_n(D_n(h_{(n)})|h_{(n)}) = 1$ for all $h_{(n)}$ in dom D_n. Since D_n has closed values, π_n is feasible if and only if $\text{supp}(\pi_n(h_{(n)})) \subseteq D_n(h_{(n)})$, each $h_{(n)}$ in dom D_n. The set of all feasible date n strategies is denoted by Π_n.

Remark (4.1.2). For each n, $\Pi_n \neq \emptyset$. To see this, let $\bar{D}_n = D_n$ on dom D_n and $\bar{D}_n = A_n$, otherwise. It is easy to check that the correspondence $\bar{D}_n$ has a measurable graph and σ-compact values. By [6: Theorem 1], there exists a $\mathbb{B}(H_{(n)})$- measurable function $\pi_n: H_{(n)} \to A_n$ that selects from $\bar{D}_n$. Identifying π_n with the function whose value at $h_{(n)}$ is the probability measure degenerate at $\pi_n(h_{(n)})$ makes π_n a transition probability and clearly then $\pi_n \in \Pi_n$.

<u>Definitions</u> (4.1.3). A (<u>feasible</u>) <u>date n plan</u> is a sequence $\pi^n = \{\pi_m\}_{m \geq n}$, where π_m is a (feasible) date m strategy for each $m \geq n$. The set of feasible date n plans is denoted by Π^n. A date 1 (feasible) plan is referred to simply as a (feasible) plan, and Π denotes the set of feasible plans.

Given the agent's opinion $q = \{q_n\}_{n \in \mathbb{N}}$ and a date n plan $\pi^n = \{\pi_m\}_{m \geq n}$, there is a transition probability $\pi^n q \equiv \pi_n q_n \pi_{n+1} q_{n+1} \cdots$ from $H_{(n)}$ to $H^{(n)}$ given by the standard (conditional) product measure theorem [2: Theorem 2.7.2]. Similarly, for $n \geq 2$ there is a transition probability $q\pi^n \equiv q_{n-1}\pi_n q_n \pi_{n+1} \cdots = q_{n-1}\pi^n q$ from $H_{(n-1)} \times A_{n-1}$ to $S_n \times H^{(n)}$. For a plan π, we let $\pi q = \pi_1 q_1 \pi_2 q_2 \cdots$.

<u>Expected Utility Maximization</u> (4.1.4). The infinite horizon choice problem for the typical agent satisfying (A.1) - (A.4) is to find $\pi^* \in \Pi$, if it exists, such that

$$\int_{H^{(1)}} u(\mathrm{proj}_C(h^{(1)}))\pi q(dh^{(1)} | h_{(1)}) \leq \int_{H^{(1)}} u(\mathrm{proj}_C(h^{(1)}))\pi^* q(dh^{(1)} | h_{(1)}),$$

for every $h_{(1)}$ in H_1^* and every π in Π, where proj_C denotes the projection of $H^{(1)}$ onto the set C of consumption histories and u is the utility function of the agent given in (A.2). In words, the agent seeks a feasible plan that maximizes his expected utility (the integral above) for each initial date price system in Q_1. This last restriction of the problem arises because of the characterization of Theorem (3.2.2).

4.2. Attainable Futures: Continuity and Compactness

In Section 5, we show that the problem of (4.1.4) does indeed have a solution and that the corresponding demand relations for this solution have the regularity properties needed for the equilibrium analysis of [29]. To prepare the way for this solution, we consider an important relation between histories and posterities at a given date.[10]

Definitions (4.2.1). For each n in $\mathbb{N}$, let $\mathcal{F}_n : H_{(n)} \times A_n \Rightarrow \mathcal{P}(S_{n+1} \times H^{(n+1)})$ be defined by

$$\mathcal{F}_n(h_{(n)}, a_n) = \{\nu \in \mathcal{P}(S_{n+1} \times H^{(n+1)}) : \nu = q\pi^{n+1}(h_{(n)}, a_n),\ \pi^{n+1} \in \Pi^{n+1}\}.$$

The relation $\mathcal{F}_n$ associates with a history $(h_{(n)}, a_n)$ through date n the set of probability measures on posterities at $(h_{(n)}, a_n)$ that are attainable by following a feasible date n + 1 plan. The elements of $\mathcal{F}_n(h_{(n)}, a_n)$ are referred to as attainable futures. The relation $\mathcal{F}_n$ is analogous to Jordan's future decision rule relation [23: 2.5] and the relation F^n of [24: (2.6)] was modeled on $\mathcal{F}_n$ with the current action variable included as part of the future at n.

As we show in Section 5, at each date n and history $h_{(n)}$ in dom D_n, the agent can be viewed as selecting for each feasible action a_n an attainable future ν in $\mathcal{F}_n(h_{(n)}, a_n)$ to maximize expected utility conditional on $(h_{(n)}, a_n)$, and then choosing a_n in $D_n(h_{(n)})$ to maximize this maximum expected utility conditional on $h_{(n)}$. The agent's demand relation is derived from this second stage of optimization.

Application of the usual maximum theorems is crucial in this two stage optimization process. The regularity required for the first stage is given in the following theorem. The proof of this result is involved and is given in Section 4.4.

Theorem (4.2.2). dom $\mathcal{F}_n = H_{(n)} \times A_n$, and $\mathcal{F}_n|\Delta_n$ is continuous and compact valued.

4.3. Attainable Futures: Convexity and Monotonicity

To obtain concavity and monotonicity of the derived utility function of an agent, it is necessary to have a particular convexity property and a monotonicity property of the correspondence $\mathcal{F}_n$. The convexity property, the prototype of which is due to Jordan [21: Proposition 2.33], involves the following setting. For $i = 1,2$, let $\delta_n^i = (h_{(n)}, a_n^i)$ be an element of Δ_n, and let ν_i be in $\mathcal{F}_n(\delta_n^i)$, $\nu_i = q_n \pi_{n+1}^i q_{n+1} \pi_{n+2}^i \ldots (\delta_n^i)$, where $\pi_{n+k}^i \in \Pi_{n+k}$ for each k in $\mathbb{N}$. For β given $0 < \beta < 1$, let $\delta_n = \beta\delta_n^1 + (1 - \beta)\delta_n^2$. It follows from Lemma (3.2.1) that δ_n belongs to Δ_n. It is not the case necessarily, however, that $\beta\nu^1 + (1 - \beta)\nu^2$ belongs to $\mathcal{F}_n(\delta_n)$, i.e., the graph of $\mathcal{F}_n|\Delta_n$ need not be convex.

There is, however, a method of combining ν^1 and ν^2 so that the resulting combination belongs to $\mathcal{F}_n(\delta_n)$ and this combination is all that is needed to establish concavity of the derived utility function (Lemma (5.3.1)).

Notation and Definitions (4.3.1). Let $\hat{H}_1 = H_1$ and for $n \geq 2$, let $\hat{H}_n = A_{n-1} \times A_{n-1} \times S_n$. Then as in Section 2.1, $\hat{H}_{(n)} = \hat{H}_1 \times \ldots \times H_n$ and $\hat{H}^{(n)} = \hat{H}_{n+1} \times \hat{H}_{n+2} \times \ldots$. If $0 < \beta < 1$, let $\hat{\beta}_n\colon\ S_{n+1} \times \hat{H}^{(n+1)} \to S_{n+1} \times H^{(n+1)}$ be defined by

$$\hat{\beta}_n(s_{n+1}, \hat{h}^{(n+1)}) = (s_{n+1}, \beta a_{n+1}^1 + (1 - \beta)a_{n+1}^2, s_{n+2}, \beta a_{n+2}^1 + (1 - \beta)a_{n+2}^2, \ldots),$$

where $\hat{h}^{(n+1)} = (a_{n+1}^1, a_{n+1}^2, s_{n+2}, a_{n+2}^1, a_{n+2}^2, \ldots)$, with a_{n+k}^i in A_{n+k}, all k.

Since A_{n+k} is convex, $\hat{\beta}_n$ is well defined and is clearly continuous on $S_{n+1} \times \hat{H}^{(n+1)}$. For each k in $\mathbb{N}$, let $\hat{H}^{(n)}_{(n+k)} = S_{n+1} \times \hat{H}_{n+2} \times \cdots \times \hat{H}_{n+k}$ with $\hat{H}^{(n)}_{(n+1)} = S_{n+1}$, and define $\hat{\pi}_{n+k}: \hat{H}^{(n)}_{(n+k)} \to \mathcal{P}(A_{n+k} \times A_{n+k})$ by

$$\hat{\pi}_{n+k}(s_{n+1}, a^1_{n+1}, a^2_{n+1}, s_{n+2},\ldots,s_{n+k})$$

$$= \pi^1_{n+k}(\delta^1_n, s_{n+1}, a^1_{n+1}, s_{n+2},\ldots,s_{n+k}) \times \pi^2_{n+k}(\delta^2_n, s_{n+1}, a^2_{n+1}, s_{n+2},\ldots,s_{n+k}),$$

where "×" on the right here denotes the classical product measure [2: Theorem (2.6.2)]. The function $\hat{\pi}_{n+k}$ so defined is a transition probability from $\hat{H}^{(n)}_{(n+k)}$ to $A_{n+k} \times A_{n+k}$. We obtain a measure $\hat{\nu}$ on $S_{n+1} \times \hat{H}^{(n+1)}$ by piecing the $\hat{\pi}_{n+k}$ together (using the conditional product measure theorem [2: Theorem (2.7.2)]) with the q_{n+k} evaluated at δ_n, i.e., $\hat{\nu} = q_n\hat{\pi}_{n+1}q_{n+1}\hat{\pi}_{n+2}\cdots(\delta_n) \in \mathcal{P}(S_{n+1} \times \hat{H}^{(n+1)})$. Finally, the map $\hat{\beta}_n$ can be used to induce $\hat{\nu}$ onto $S_{n+1} \times H^{(n+1)}$, i.e., define $\nu(B) = \hat{\nu}(\hat{\beta}^{-1}_n(B))$, for each B in $\mathcal{B}(S_{n+1} \times H^{(n+1)})$.

Intuitively, one can think of the actions in $\hat{H}^{(n+1)}$ as being selected by independent date n + k strategies π^1_{n+k} and π^2_{n+k} and the prices as being selected by the opinions in the usual sense. The resulting probability law on sequences of price - action - action triples is $\hat{\nu}$. The induced measure ν in essence represents the selection of actions that are β-convex combinations of the independently chosen actions in $\hat{H}^{(n+1)}$. The convexity result states that the measure ν is an attainable future at δ_n.

Lemma (4.3.2). If $0 < \beta < 1$, $\delta_n^i = (h_{(n)}, a_n^i) \in \Delta_n$ and $\nu^i \in \mathcal{F}_n(\delta_n^i)$, $i = 1,2$, then for $\delta_n = \beta\delta_n^1 + (1-\beta)\delta_n^2$ and ν defined in (4.3.1), $\nu \in \mathcal{F}_n(\delta_n)$.

Proof: Let β, δ_n^i, and ν^i, $i = 1,2$, be as hypothesized, and let ν be defined as in (4.3.1). Let ν_{n+k}^i and ν_{n+k} denote the marginals of ν for $H_{(n+k)}^{(n)} = S_{n+1} \times A_{n+1} \times \ldots \times S_{n+k}$ and $H_{(n+k)}^{(n)} \times A_{n+k}$, respectively, $k = 1,2,\ldots.$ Also for each k let π_{n+k} be a regular conditional distribution [5: Theorem 4.34] of ν for (the projection onto) A_{n+k} given (the projection onto) $H_{(n+k)}^{(n)}$. Using the properties of these distributions one can show by induction that $\nu = q_n\pi_{n+1}q_{n+1}\pi_{n+2}\ldots(\delta_n)$, and that for each k in $\mathbb{N}$, $\nu_{n+k}^1(H_{n+k}^*(\delta_n)) = 1 = \nu_{n+k}(\Delta_{n+k}(\delta_n))$. It follows that the set

$$B_k = \{h_{(n+k)}^{(n)}: (\delta_n, h_{(n+k)}^{(n)}) \in \text{dom } D_{n+k}, \pi_{n+k}(D_{n+k}(\delta_n, h_{(n+k)}^{(n)}) | h_{(n+k)}^{(n)}) < 1\}$$

belongs to $\mathbb{B}(H_{(n+k)}^{(n)})$ and has ν_{n+k}^1 - measure zero. Then as in Lemma (4.4.8), π_{n+k} can be modified and extended to an element of Π_{n+k} so that the representation $\nu = q_n\pi_{n+1}q_{n+1}\pi_{n+2}\ldots(\delta_n)$ remains valid. It follows that $\nu \in \mathcal{F}_n(\delta_n)$.□

The desired monotonicity property is more easily stated.

Lemma (4.3.3). If $\delta_n^i = (h_{(n)}, a_n^i) \in \Delta_n$, $i = 1,2$, and $a_n^1 \leqq a_n^2$, then $\mathcal{F}_n(\delta_n^1) \subseteq \mathcal{F}_n(\delta_n^2)$.

Proof: Let $\delta_n^i = (h_{(n)}, a_n^i)$, $i = 1,2$, be as hypothesized, and let ν^1 be in $\mathcal{F}_n(\delta_n^1)$, $\nu^1 = q_n\pi_{n+1}^1 q_{n+1}\pi_{n+2}^1\ldots(\delta_n^1)$ where $\pi_{n+k}^1 \in \Pi_{n+k}$, $k = 1,2,\ldots.$ Define π_{n+1}^2 by

$$\pi^2_{n+1}(\delta^1_n,s_{n+1}) = \pi^1_{n+1}(\delta^1_n,s_{n+1}), \quad \delta_n = \delta^2_n, s_{n+1} \in H^*_{n+1}(\delta^1_n),$$

$$= \pi^1_{n+1}(\delta^2_n,s_{n+1}), \quad \delta_n = \delta^2_n, s_{n+1} \notin H^*_{n+1}(\delta^1_n)$$

$$= \pi^1_{n+1}(\delta_n,s_{n+1}), \quad \delta_n \neq \delta^2_n.$$

Then π^2_{n+1} is a transition probability from $H_{(n+1)}$ to A_{n+1} since π^1_{n+1} is. Also for $s_{n+1} \in H^*_{n+1}(\delta^1_n) = H^*_{n+1}(\delta^2_n)$, $a^1_n \lneq a^2_n$ implies that $D_{n+1}(\delta^1_n,s_{n+1}) \subseteq D_{n+1}(\delta^2_n,s_{n+1})$. Since $\pi^1_{n+1} \in \Pi_{n+1}$, it follows that $\pi^2_{n+1} \in \Pi_{n+1}$.

For $k > 1$, define π^2_{n+k} by

$$\pi^2_{n+k}(\delta_n,h^{(n)}_{(n+k)}) = \pi^1_{n+k}(\delta^1_n,h^{(n)}_{(n+k)}), \qquad \delta_n = \delta^2_n$$

$$= \pi^1_{n+k}(\delta_n,h^{(n)}_{(n+k)}), \qquad \delta_n \neq \delta^2_n.$$

Since D_{n+k} does not vary with δ_n for $h^{(n)}_{(n+k)}$ fixed, we have that $\pi^2_{n+k} \in \Pi_{n+k}$.

For these definitions, it follows that for every s_{n+1} in $H^*_{n+1}(\delta^i_n)$, $\pi^2_{n+1}q_{n+1}\pi^2_{n+2}q_{n+2}\ldots(\delta^2_n,s_{n+1}) = \pi^1_{n+1}q_{n+1}\pi^1_{n+2}q_{n+2}\ldots(\delta^1_n,s_{n+1})$. Clearly for $\nu^2 = q_n\pi^2_{n+1}q_{n+1}\pi^2_{n+2}\ldots(\delta^2_n)$, $\nu^2 \in \mathcal{F}_n(\delta^2_n)$, and since $q_n(\delta^1_n) = q_n(\delta^2_n)$, it follows that $\nu^1 = \nu^2$.□

Intuitively, if $a^1_n \lneq a^2_n$, then $r_{n+1}(h_{(n)},a^1_n,s_{n+1}) \leq r_{n+1}(h_{(n)},a^2_n,s_{n+1})$ for each $s_{n+1} \in \mathbb{R}^{2\ell}_+$, and consequently the set of actions feasible at date n+1 following $(h_{(n)},a^2_n)$ is at least as large as the set following $(h_{(n)},a^1_n)$.

Since the feasible action sets beyond date n + 1 are unaffected by the action at date n, it seems intuitive that any future attainable from $(h_{(n)}, a_n^1)$ would also be attainable from $(h_{(n)}, a_n^2)$, and this is precisely what Lemma (4.3.3) states.

Based on this argument, one would expect $\mathcal{F}_{n+1}$ to be independent of the action taken at date n. The final result of this section states this independence formally.

Lemma (4.3.4). Let $\delta_n^i = (h_{(n)}, a_n^i)$ be in Δ_n, $i = 1,2$, and suppose $s_{n+1} \in \text{int}\ \sigma_n(\delta_n^i)$. If $h^i_{(n+1)} = (\delta_n^i, s_{n+1})$, $i = 1,2$, and if $a_{n+1} \in D_{n+1}(h^1_{(n+1)}) \cap D_{n+1}(h^2_{(n+1)})$, then $\mathcal{F}_{n+1}(h^1_{(n+1)}, a_{n+1}) = \mathcal{F}_{n+1}(h^2_{(n+1)}, a_{n+1})$.

Proof: Let $\delta^i_{n+1} = (h^i_{(n+1)}, a_{n+1})$, where $h^i_{(n+1)}$ and a_{n+1} are as hypothesized, $i = 1,2$. Let $\nu^1 = q_{n+1}\pi^1_{n+2}q_{n+2}\pi^1_{n+3}\cdots(\delta^1_{n+1})$ be in $\mathcal{F}_{n+1}(\delta^1_{n+1})$. For each $k \geq 2$, let $\pi^2_{n+k}(\delta^2_{n+1},\cdot) = \pi^1_{n+k}(\delta^1_{n+1},\cdot)$ and $\pi^2_{n+k}(\delta_{n+1},\cdot) = \pi^1_{n+k}(\delta_n,\cdot)$, all $\delta_{n+1} \neq \delta^2_{n+1}$. Since $D_{n+k}(\delta^1_{n+1},\cdot) = D_{n+k}(\delta^2_{n+1},\cdot)$ for $k \geq 2$, it follows that $\pi^2_{n+k} \in \Pi_{n+k}$ for $k \geq 2$, and that $\nu^2 \equiv q_{n+1}\pi^2_{n+2}q_{n+2}\pi^2_{n+3}\cdots(\delta^2_{n+1}) = \nu^1$. Since the argument is symmetric in δ^1_{n+1} and δ^2_{n+2}, the result follows.□

4.4 Proof of Theorem (4.2.2)

First note that dom $\mathcal{F}_n = H_{(n)} \times A_n$ by Remark (4.1.2). It remains to show that $\mathcal{F}_n$ restricted to Δ_n is continuous and compact valued. This will be done through a series of lemmas.

Definition (4.4.1) For each n, let E_n: $H_{(n)} \times A_n \Rightarrow S_{n+1} \times H^{(n+1)}$ be defined by

$$E_n(h_{(n)},a_n) = \{(s_{n+1},h^{(n+1)}): \alpha_{n+k}(h^{(n+1)}) \in D_{n+k}(h_{(n)},a_n,s_{n+1},\ldots,\xi_{n+k}(h^{(n+1)})), k \in N\}$$

Then $E_n(h_{(n)},a_n)$ is the set of posterities at date n and history $(h_{(n)},a_n)$. (C.f. [23: 2.4]).

Lemma (4.4.2). $\Delta_n \subseteq \text{dom } E_n$, E_n has a closed graph in H, and $\text{supp}(\nu) \subseteq E_n(\delta_n)$ if $\delta_n \in \Delta_n$ and $\nu \in \mathcal{F}_n(\delta_n)$.

Proof: The first two statements follow from Lemma (3.2.1). To prove the last statement, suppose that $\delta_n \in \Delta_n$ and that $\nu \in \mathcal{F}_n(\delta_n)$, $\nu = q\pi^{n+1}(\delta_n)$, some $\pi^{n+1} = \{\pi_m\}_{m \geq n+1}$ in Π^{n+1}. Suppose also that $(s_{n+1},h^{(n+1)}) \in \text{supp}(\nu)$ and let $U \subseteq S_{n+1} \times H^{(n+1)}$ be an open set containing $(s_{n+1},h^{(n+1)})$. We may take $U = V^1_{n+1} \times V^2_{n+1} \times \ldots \times V^1_t \times V^2_t \times S_{t+1} \times H^{(t+1)}$, for some $t > n+1$, V^1_k open in S_k and V^2_k open in A_k, $k = n+1,\ldots,t$. Let $\hat{V}^1_{n+1} = \{s_{n+1}: {}^{\pi^{n+1}}q(U(s_{n+1})|\delta_n,s_{n+1}) > 0\}$. Since $\nu(U) > 0$, it follows that $U^1_{n+1} \equiv \hat{V}^1_{n+1} \cap V^1_{n+1} \cap \sigma_n(\delta_n) \neq \emptyset$. Let $\bar{s}_{n+1}$ be in U^1_{n+1} and let $\hat{V}^2_{n+1} = \{a_{n+1}: q\pi^{n+2}(U(\bar{s}_{n+1},a_{n+1})|\delta_n,\bar{s}_{n+1},a_{n+1}) > 0\}$. Then $U^2_{n+1} \equiv \hat{V}^2_{n+1} \cap V^2_{n+1} \cap \text{supp}(\pi_{n+1}(\delta_n,\bar{s}_{n+1})) \neq \emptyset$. Let $\bar{a}_{n+1}$ be in U^2_{n+1}. Continuing in this way, one constructs a sequence $(\bar{s}_{n+1},\bar{a}_{n+1},\ldots,\bar{s}_t,\bar{a}_t)$ in $V^1_{n+1} \times V^2_{n+1} \times \ldots \times V^1_t \times V^2_t$ such that $\bar{s}_k$ is in $\sigma_{k-1}(\delta_n,\ldots,\bar{s}_{k-1},\bar{a}_{k-1})$ and $\bar{a}_k$ is an element of $\mathrm{supp}(\pi_k(\delta_n,\bar{s}_{n+1},\ldots,\bar{s}_k))$, $k = n+1,\ldots,t$. For $k > t$, choose $\bar{s}_k$ in $\sigma_{k-1}(\delta_n,\bar{s}_{n+1},\ldots,\bar{s}_{k-1},\bar{a}_{k-1})$ and $\bar{a}_k$ in $\mathrm{supp}(\pi_k(\delta_n,\bar{s}_{n+1},\ldots,\bar{s}_k))$ arbitrarily. Let $(\bar{s}_{n+1},\bar{h}^{(n+1)}) = (\bar{s}_{n+1},\bar{a}_{n+1},\ldots.)$. Then $(\bar{s}_{n+1}),\bar{h}^{(n+1)}) \notin U \cap E_n(\delta_n)$. Thus every open set containing $(s_{n+1},h^{(n+1)})$ contains a point of $E_n(\delta_n)$ and $(s_{n+1},h^{(n+1)})$ is a point of closure of $E_n(\delta_n)$. □

The next three results deal with specific properties of opinions and with essential sets of finite dimensional distributions of measures of the form $q\pi^{n+1}(\cdot)$ formed from feasible plans.

Lemma (4.4.3). If $\delta_n \in H_{(n)} \times A_n$, there is a sequence $\{K_m\}_{m\in\mathbb{N}}$ of non-empty convex and relatively compact subsets of $\mathbb{R}^{2\ell}$ such that $\mathrm{cl}\ K_m \subseteq K_{m+1}$ for each m and $\bigcup_{m\in\mathbb{N}} K_m = \mathrm{int}\ \sigma_n(\delta_n)$.

Proof: By [12: Theorem 7.2, p. 241], $\mathrm{int}\ \sigma_n(\delta_n) = \bigcup_{m\in\mathbb{N}} U_m$, where U_m is non-empty, open and relatively compact. Let $K_m = \mathrm{co}\ U_m$ and the result follows. □

Lemma (4.4.4). If $\delta_n \in H_{(n)} \times A_n$ and $\varepsilon > 0$, there is a compact set $K \subseteq \mathbb{R}^{2\ell}$ and an open set $U \subseteq H_{(n)}$ such that $\delta_n \in U \times A_n$ and if $\delta'_n \in U \times A_n$, then $K \subseteq \mathrm{int}\ \sigma_n(\delta'_n)$ and $q_n(K|\delta'_n) > 1 - \varepsilon$.

Proof: By Lemma (4.4.3), there is a compact set $K \subseteq \mathrm{int}\ \sigma_n(\delta_n)$ such that $q_n(\mathrm{int}\ K|\delta_n) > 1 - \varepsilon/2$. By continuity of q_n, there is an open set $\hat{U} \subseteq H_{(n)}$ such that $\delta_n \in \hat{U} \times A_n$ and if $\delta'_n \in \hat{U} \times A_n$, then $q_n(\mathrm{int}\ K|\delta'_n) > q_n(\mathrm{int}\ K|\delta_n) - \varepsilon/2 > 1 - \varepsilon$.

Since Σ_n^o is an open set (Remark (2.3.8)), if $s \in K$, there exists a pair of open sets (V,W) with $V \subseteq H_{(n)}$ and $W \subseteq \mathbb{R}^{2\ell}$ such that $\delta_n \in V \times A_n$, $s \in W$, and if $\delta'_n \in V \times A_n$, then $W \subseteq \text{int}\ \sigma_n(\delta'_n)$. Since the W cover K, there is a finite number of points $s^1,\ldots,s^m$ in K and corresponding pairs $(V_1,W_1),\ldots,(V_m,W_m)$ such that $K \subseteq \bigcup_{i=1}^m W_i \equiv W$. For $V = \bigcap_{i=1}^m V_i$, $V \subseteq H_{(n)}$ is open, $\delta_n \in V \times A_n$, and if $\delta'_n \in V \times A_n$, then $K \subseteq W \subseteq \text{int}\ \sigma_n(\delta'_n)$. Let $U = \hat{U} \cap V$, and the proof is complete.□

Lemma (4.4.5). If $\pi = \{\pi_n\}_{n\in\mathbb{N}} \in \Pi$, for each n in $\mathbb{N}$ and $k \geq 0$, if $\delta_n \in \Delta_n$, then

$$(4.4.6)\qquad q_n\pi_{n+1}\cdots\pi_{n+k}q_{n+k}(H^*_{n+k+1}(\delta_n)|\delta_n) = 1;$$

$$(4.4.7)\qquad q_n\pi_{n+1}\cdots q_{n+k}\pi_{n+k+1}(\Delta_{n+k+1}(\delta_n)|\delta_n) = 1.$$

Proof: By definition, if $\delta_n \in \Delta_n$, then $H^*_{n+1}(\delta_n) = \text{int}\ \sigma_n(\delta_n)$, and (4.4.6) for $k = 0$ follows from assumption (A.3)(iv). Also, if $s_{n+1} \in \text{int}\ \sigma_n(\delta_n)$, then $(\delta_n,s_{n+1}) \in H^*_{n+1} \subseteq \text{dom}\ D_{n+1}$, and hence $\pi_{n+1}(\Delta_{n+1}(\delta_n,s_{n+1})|\delta_n,s_{n+1}) = 1$. Thus

$$q_n\pi_{n+1}(\Delta_{n+1}(\delta_n)|\delta_n) = \int_{\text{int}\ \sigma_n(\delta_n)}\pi_{n+1}(\Delta_{n+1}(\delta_n,s_{n+1})|\delta_n,s_{n+1})q_n(ds_{n+1}|\delta_n)$$

$$= q_n(\text{int}\ \sigma_n(\delta_n)|\delta_n) = 1,$$

and (4.4.7) holds for $k = 0$.

Assume (4.4.6) and (4.4.7) hold for some $k > 0$. If $(s_{n+1},\ldots,a_{n+k+1}) \in \Delta_{n+k+1}(\delta_n)$, then $H^*_{n+k+2}(\delta_n,s_{n+1},\ldots,a_{n+k+1}) = \text{int}\ \sigma_n(\delta_n,s_{n+1},\ldots,a_{n+k+1})$. Again by (A.3)(iv), (4.4.7) for k implies (4.4.6) holds for $k + 1$. By an argument identical to the one for $k = 0$, if (4.4.6) holds for $k + 1$, then so does (4.4.7).□

The strategy in proving that $\mathcal{F}_n$ is u.h.c. and compact valued on Δ_n involves showing that $\mathcal{F}_n$ restricted to Δ_n has a closed graph and then showing that from $\mathcal{F}_n$ over a convergent sequence from Δ_n, only tight sequences of measures can be selected. The result then follows from Prohorov's Theorem.

<u>Lemma</u> (4.4. 8). $\mathcal{F}_n|_{\Delta_n}$ has a closed graph.

Proof: Assume that (δ_n, ν) is a point of closure of the graph of $\mathcal{F}_n|_{\Delta_n}$. Then there is a sequence $\{(\delta_n^k, \nu^k)\}_{k\in\mathbb{N}}$ such that for each k, $\delta_n^k \in \Delta_n$ and $\nu^k \in \mathcal{F}_n(\delta_n^k)$, and such that $\lim_k \delta_n^k = \delta_n$ and $\lim_k \nu^k = \nu$. For each k, let $\{\pi_r^k\}_{r\geq n+1}$ denote an element of Π^{n+1} such that $\nu^k = m^k(\delta_n^k) \equiv q_n\pi_{n+1}^k q_{n+1}\pi_{n+2}^k \cdots (\delta_n^k)$. Since Δ_n is closed, it suffices to show that there exists $\{\pi_r\}_{r\geq n+1}$ in Π^{n+1} such that $\nu = q_n\pi_{n+1}q_{n+1}\pi_{n+2}\cdots(\delta_n)$.

For each positive integer j, let ν_{n+j}^1 and ν_{n+j} denote the marginals of ν for $H_{(n+j)}^{(n)} = S_{n+1} \times A_{n+1} \times \ldots \times S_{n+j}$ and $H_{(n+j)}^{(n)} \times A_{n+j}$, respectively. Also, let π_{n+j} be a regular conditional distribution for (the projection onto) A_{n+j} given (the projection onto) $H_{(n+j)}^{(n)}$, [5: Theorem 4.34], i.e., π_{n+j} is a transition probability from $H_{(n+j)}^{(n)}$ to A_{n+j} satisfying

$$\nu_{n+j}(B_1 \times B_2) = \int_{B_1} \pi_{n+j}(B_2 | h_{(n+j)}^{(n)}) \nu_{n+j}^1 (dh_{(n+j)}^{(n)}),$$

for all B_1 in $\mathbb{B}(H_{(n+j)}^{(n)})$ and B_2 in $\mathbb{B}(A_{n+j})$.

By continuity of projections, $m^k(\delta_n^k) \to \nu$ implies that for each j, $q_n\pi_{n+1}^k \cdots \pi_{n+j-1}^k q_{n+j-1}(\delta_n^k) \to \nu_{n+j}^1$ and $q_n\pi_{n+1}^k \cdots q_{n+j-1}\pi_{n+j}^k(\delta_n^k) \to \nu_{n+j}$. Using these results and the above formula for ν_{n+j} it can be shown by a straightforward but tedious induction argument that $\nu^1_{n+j} = q_n\pi_{n+1}\cdots\pi_{n+j-1}q_{n+j-1}(\delta_n)$ and $\nu_{n+j} = q_n\pi_{n+1}\cdots q_{n+j-1}\pi_{n+j}(\delta_n)$, for each j (here the product measures are formed using $q_n(\delta_n), q_{n+1}(\delta_n,\cdot),\ldots$, and [2: Theorem 2.7.2]).

We now use the conditional distributions π_{n+j} to construct the desired feasible date n+1 plan. Note first that if $(s_{n+1}, h^{(n+1)}) \in \operatorname{supp}(\nu)$ and if $U \subseteq S_{n+1} \times H^{(n+1)}$ is open and $(s_{n+1}, h^{(n+1)}) \in U$, then $m^k(\delta^k_n) \to \nu$ implies that $\underline{\lim}_k\, m^k(U|\delta^k_n) \geq \nu(U) > 0$. It then follows by a straightforward argument that $\operatorname{supp}(\nu) \subseteq \operatorname{Li}(\operatorname{supp}(m^k(\delta^k_n)))$, where "Li" denotes topological limit inferior [20: p. 15]. By Lemma (4.4.2), $\operatorname{supp}(m^k(\delta^k_n)) \subseteq E_n(\delta^k_n)$ and $\operatorname{supp}(\nu) \subseteq \operatorname{Li}(\operatorname{supp}(m^k(\delta^k_n))) \subseteq E_n(\delta_n)$. For each j, let $B_j = \{h^{(n)}_{(n+j)} : (\delta_n, h^{(n)}_{(n+j)}) \in \operatorname{dom} D_{n+j}, \pi_{n+j}(D_{n+j}(\delta_n, h^{(n)}_{(n+j)})|h^{(n)}_{(n+j)}) < 1\}$. Then $B_j \in \mathbb{B}(H^{(n)}_{(n+j)})$ and it follows that $\nu^1_{n+j}(B_j) = 0$ for each j.

Let $\bar{\pi}_{n+j}$ be any element of Π_{n+j}, and extend π_{n+j} be defining $\pi_{n+j}(\delta'_n, h^{(n)}_{(n+j)})$ to be $\pi_{n+j}(h^{(n)}_{(n+j)})$ if $\delta'_n = \delta_n$ and $h^{(n)}_{(n+j)} \in B^c_j$, and to be $\bar{\pi}_{n+j}(\delta'_n, h^{(n)}_{(n+j)})$ otherwise. Then clearly π_{n+j} so extended belongs to Π_{n+j}, and it is easily verified that $\nu = q_n\pi_{n+1}q_{n+1}\pi_{n+2}\cdots(\delta_n)$.□

<u>Theorem</u> (4.4.9). $\mathcal{F}_n$ is u.h.c. and compact valued on Δ_n.

Proof: Suppose that $\{\delta^k_n\}_{k\in\mathbb{N}} \subseteq \Delta_n$ with $\delta^k_n \to \delta^0_n$, and that $\nu^k \in \mathcal{F}_n(\delta^k_n)$ for each k. Let $\{\pi^k_r\}_{r\geq n+1}$ be an element of Π^{n+1} such that $\nu^k = m^k(\delta^k_n)$, where $m^k(\delta_n) \equiv q_n\pi^k_{n+1}q_{n+1}\pi^k_{n+2}\ldots(\delta_n)$. By Lemma (4.4.8) and [20: Theorem 1, p.24], the theorem will be proved if it can be shown that $\{m^k(\delta^k_n)\}_{k\in\mathbb{N}}$ has a convergent subsequence. To do this, we show that this sequence is tight.

Let $\varepsilon > 0$ be given. By Lemma (4.4.4), there is a positive integer k_0 and a compact set K_0 such that for $k \geq k_0$, $K_0 \subseteq \operatorname{int} \sigma_n(\delta^k_n)$ and $q(K_0|\delta^k_n) > 1 - \varepsilon 2^{-i}$.

Also, by Lemma (4.4. 3), for $k = 1,\ldots,k_0 - 1$, there is a compact set K_0^k such that $K_0^k \subseteq \text{int } \sigma_n(\delta_n^k)$ and $q_n(K_0^k|\delta_n^k) > 1 - \varepsilon 2^{-1}$. For $k \geq k_0$ let $K_0^k = K_0$, and let $L_1 = \cup_{k\geq 1} K_0^k$. For $k \geq 0$, let $T_1^k = \{\delta_n^k\} \times K_0^k$ and $J_1^k = \{(s_{n+1}, a_{n+1}): s_{n+1} \in K_0^k, a_{n+1} \in D_{n+1}(\delta_n^k, s_{n+1})\}$, and let $T_1 = \cup_{k\geq 0} T_1^k$ and $J_1 = \cup_{k\geq 0} J_1^k$. Then $T_1 \subseteq H_{n+1}^*$, T_1 is closed, and since $T_1 \subseteq \bar{J}_0 \times L_1$, T_1 is compact, where $\bar{J}_0 = \{\delta_n^k\}_{k\geq 0}$. By Theorem (3.2.3), $D_{n+1}(T_1)$ and J_1^k, for each k, are compact. Thus J_1 is measurable and for each $k \geq 1$, $q_n \pi_{n+1}^k (J_1|\delta_n^k) \geq q_n \pi_{n+1}^k (J_1^k|\delta_n^k) = q_n(K_0^k|\delta_n^k) > 1 - \varepsilon 2^{-1}$. Let $\bar{J}_1 = \cup_{k\geq 0}\{\delta_n^k\} \times J_1^k$. One can verify that $\bar{J}_1$ is closed, and since $\bar{J}_1 \subseteq \bar{J}_0 \times L_1 \times D_{n+1}(T_1)$, $\bar{J}_1$ is compact.

The proof now proceeds by induction. Suppose that for sime integer $r > 1$, there exist compact sets $\bar{J}_{r-1} \subseteq \Delta_{n+r-1}$, $L_r \subseteq \mathbb{R}^{2\ell}_{++}$, and $T_r \subseteq H_{n+r}^*$, and for every $k \geq 0$, there is a compact set J_r^k such that (a) $\{\delta_n^k\} \times J_r^k \subseteq \Delta_{n+r}$,
(b) $q_n \pi_{n+1}^k \cdots q_{n+r-1} \pi_{n+r}^k (J_r^k|\delta_n^k) > (1 - \varepsilon 2^{-r})(1 - \varepsilon 2^{-r+1}) \ldots (1 - \varepsilon 2^{-1})$, and
(c) $\bar{J}_r \equiv \cup_{k\geq 0}\{\delta_n^k\} \times J_r^k \subseteq \bar{J}_{r-1} \times L_r \times D_{n+r}(T_r)$ and $\bar{J}_r$ is compact.

By Lemma (4.4.4), if $\delta_{n+r} \in H_{(n+r)} \times A_{n+r}$, there is a compact set $K \subseteq \mathbb{R}^{2\ell}$ and an open set $U \subseteq H_{(n+r)}$ such that $\delta_{n+r} \in U \times A_{n+r}$ and if $\delta'_{n+r} \in U \times A_{n+r}$, then $K \subseteq \text{int } \sigma_{n+r}(\delta'_{n+r})$ and $q_{n+r}(K|\delta'_{n+r}) > 1 - \varepsilon 2^{-r-1}$. Since $H_{(n+r)}$ is locally compact [12: Theorem 6.5(4), p.239], we may assume that cl U is compact and that these properties hold with U replaced by cl U. The U cover $\bar{J}_r$, which is compact by hypothesis (c). Then there is a finite set $\{\delta_{n+r}^i: i = 1,\ldots,i^*\} \subseteq \bar{J}_r$ and corresponding open sets U_i and compact sets K_i, $i = 1,\ldots,i^*$, with $\bar{J}_r \subseteq \cup_{i=1}^{i^*} U_i$. Let $L_{r+1} = \cup_{i=1}^{i^*} K_i$, and define, for $k \geq 0$, the sets

$$T_{r+1}^{ik} = \{(h_{(n+r)}^{(n)}, a_{n+r}, s_{n+r+1}): (h_{(n+r)}^{(n)}, a_{n+r}) \in J_r^k, (\delta_n^k, h_{(n+r)}^{(n)}) \in \text{cl}U_i, s_{n+r+1} \in K_i\},$$

$$J_{r+1}^{ik} = \{(h_{(n+r+1)}^{(n)}, a_{n+r+1}): h_{(n+r+1)}^{(n)} \in T_{r+1}^{ik}, a_{n+r+1} \in D_{n+r+1}(\delta_n^k, h_{(n+r+1)}^{(n)})\}.$$

Let $\hat{T}^k_{r+1} = \cup^{i*}_{i=1} T^{ik}_{r+1}$, $T^k_{r+1} = \{\delta^k_n\} \times \hat{T}^k_{r+1}$, and let $T_{r+1} = \cup_{k\geq 0} T^k_{r+1}$ and $J_{r+1} = \cup_{k\geq 0} J^k_{r+1}$.

It is easily verified that $\hat{T}^k_{r+1}$ is closed, and since $\hat{T}^k_{r+1} \subseteq J^k_r \times L_{r+1}$, both $\hat{T}^k_{r+1}$ and T^k_{r+1} are compact. One can also verify that T_{r+1} is closed, $T_{r+1} \subseteq \bar{J}_r \times L_{r+1}$, and hence by (c), T_{r+1} is compact. Finally, by construction $T_{r+1} \subseteq H^*_{n+r+1}$. By Theorem (3.2.3), $D_{n+r+1}(T_{r+1})$ and J^k_{r+1}, each $k \geq 0$, are compact. Thus J_{r+1} is measurable and for each $k \geq 1$,

$$\begin{aligned}
q_n\pi^k_{n+1}\cdots\pi^k_{n+r}q_{n+r}\pi^k_{n+r+1}(J_{r+1}|\delta^k_n) &\geq q_n\pi^k_{n+1}\cdots\pi^k_{n+r}q_{n+r}\pi^k_{n+r+1}(J^k_{r+1}|\delta^k_n)\\
&= q_n\pi^k_{n+1}\cdots\pi^k_{n+r}q_{n+r}(\hat{T}^k_{r+1}|\delta^k_n)\\
&\geq (1-\varepsilon 2^{-r-1})q_n\pi^k_{n+1}\cdots\pi^k_{n+r}(J^k_r|\delta^k_n)\\
&> (1-\varepsilon 2^{-r-1})(1-\varepsilon 2^{-r})\cdots(1-\varepsilon 2^{-1}),
\end{aligned}$$

where the last inequality follows from (b) of the induction hypothesis. Let $\bar{J}_{r+1} = \cup_{k\geq 0}\{\delta^k_n\} \times J^k_{r+1}$. It can be shown that $\bar{J}_{r+1}$ is a closed subset of Δ_{n+r+1}. Since $\bar{J}_{r+1} \subseteq \bar{J}_r \times L_{r+1} \times D_{n+r+1}(T_{r+1})$, it follows that $\bar{J}_{r+1}$ is compact.

The induction hypothesis thus holds for $r + 1$ and hence for every r in $\mathbb{N}$. Note that by definition of $\bar{J}_r$, the set J_r is compact and $J_{r+1} \subseteq J_r \times L_{r+1} \times D_{n+r+1}(T_{r+1})$, for each r. The set $J_{(r)} \equiv J_r \times S_{n+r+1} \times H^{(n+r+1)}$ is thus closed in $S_{n+1} \times H^{(n+1)}$ and $J_{(r+1)} \subseteq J_{(r)}$. Thus, $J \equiv \cap_{r\in\mathbb{N}} J_{(r)}$ is closed and $J \subseteq L_1 \times D_{n+1}(T_1) \times L_2 \times D_{n+2}(T_2) \times \ldots$, implying that J is compact. But for $k \geq 1$,

$$\begin{aligned}
m^k(J|\delta^k_n) &= \lim_r m^k(J_{(r)}|\delta^k_n)\\
&= \lim_r q_n\pi^k_{n+1}\cdots q_{n+r-1}\pi^k_{n+r}(J_r|\delta^k_n)\\
&\geq \lim_r (1-\varepsilon 2^{-r})(1-\varepsilon 2^{-r+1})\cdots(1-\varepsilon 2^{-1})
\end{aligned}$$

$$\geq 1 - \lim_r \Sigma_{i=1}^{r} \varepsilon 2^{-i}$$

$$= 1 - \varepsilon.$$

Since ε was arbitrary, the sequence $\{m^k(\delta_n^k)\}_{k\in\mathbb{N}}$ is tight. By Prohorov's Theorem [4: Theorem 6.1], $\{m^k(\delta_n^k)\}_{k\in\mathbb{N}}$ is relatively compact, and the proof is complete.□

As in [23: 5.3 - 5.7], the strategy in proving that $\mathcal{F}_n$ is l.h.c. on Δ_n involves showing that $\mathcal{F}_n$ is generated by continuous selections. Here, however, continuity of selections is limited by the restricted continuity of D_n.

<u>Lemma</u> (4.4.10). If $\varepsilon > 0$, $\pi_n \in \Pi_n$, and $\nu \in \mathcal{P}(H_{(n)})$ with $\nu(H_n^*) = 1$, then there exists π_n^* in Π_n such that $\pi_n^*|_{H_n^*}$ is continuous and for every bounded continuous function $g\colon H_{(n)} \times A_n \to \mathbb{R}$, $|\int g d\nu\pi_n^* - \int g d\nu\pi_n| < 2\varepsilon \|g\|$, where $\|g\| = \text{supp}\{|g(\delta_n)| : \delta_n \in H_{(n)} \times A_n\}$.

Proof: Let ε, π_n, and ν be as hypothesized. By Lemma (A.III), there is a closed set $B' \subseteq H_{(n)}$ such that $\nu(B') > 1 - \varepsilon$ and $\pi_n|_{B'}$ is continuous. For $B = B' \cap H_n^*$, B is a closed subset of H_n^*, $\nu(B) = \nu(B')$, and $\pi_n|_B$ is continuous. Define $\mathcal{D}_n(h_{(n)}) = \{\mu \in \mathcal{P}(A_n) : \text{supp}(\mu) \subseteq D_n(h_{(n)})\}$. By Lemma (A.IV), $\text{dom}\,\mathcal{D}_n = \text{dom}\,D_n$ and $\mathcal{D}_n|_{H_n^*}$ is continuous, compact and convex valued. Since $\pi_n|_B$ is a continuous selection for $\mathcal{D}_n|_B$, it follows from [28: Theorem 1.5(a)] that $\pi_n|_B$ can be extended to a continuous selection $\hat{\pi}_n$ for $\mathcal{D}_n|_{H_n^*}$. Let $\pi_n^* = \hat{\pi}_n$ on H_n^* and $\pi_n^* = \pi_n$, otherwise. Since $H_n^* \in \mathcal{B}(H_{(n)})$, $\pi_n^* \in \Pi_n$ and $\pi_n^*|_{H_n^*}$ is continuous.

Finally, if $g: H_{(n)} \times A_n \to \mathbb{R}$ is bounded and continuous, then

$$\left|\int g d\nu\pi_n^* - \int g d\nu\pi_n\right| = \left|\int_{B^c \times A_n} g d\nu\pi_n^* - \int_{B^c \times A_n} g d\nu\pi_n\right|$$

$$\leq 2\|g\|\, \nu(B^c) < 2\|g\|\, \varepsilon,$$

where B^c is the complement in $H_{(n)}$ of B.□

Definition (4.4.11). For each n, let $\Pi_n^* = \{\pi_n : \pi_n \in \Pi_n, \pi_n|_{H_n^*}$ is continuous$\}$, and let $\Pi^{*n} = \{\pi^n = \{\pi_m\}_{m\geq n} : \pi_m \in \Pi_m^*, m\geq n\}$. Analogous to (4.2.1), let $\mathcal{F}_n^*: H_{(n)} \times A_n \Rightarrow \mathcal{P}(S_{n+1} \times H^{(n+1)})$ be defined by

$$\mathcal{F}_n^*(h_{(n)}, a_n) = \{\nu \in \mathcal{P}(S_{n+1} \times H^{(n+1)}) : \nu = q\pi^{n+1}(h_{(n)}, a_n), \pi^{n+1} \in \Pi^{*n+1}\}.$$

By Lemma (4.4.10), $\Pi_n^* \neq \emptyset$ for each n, and consequently dom $\mathcal{F}_n^* = H_{(n)} \times A_n$.

The next result shows that $\mathcal{F}_n^*$ is dense in $\mathcal{F}_n$ on Δ_n.

Lemma (4.4.12). If $\delta_n \in \Delta_n$, then $\mathcal{F}_n(\delta_n) = \mathrm{cl}\ \mathcal{F}_n^*(\delta_n)$.

Proof: By Theorem (4.4.9), if $\delta_n \in \Delta_n$, then $\mathrm{cl}\ \mathcal{F}_n^*(\delta_n) \subseteq \mathcal{F}_n(\delta_n)$. Suppose then that $\nu \in \mathcal{F}_n(\delta_n)$, $\nu = q_n\pi_{n+1}q_{n+1}\pi_{n+2}\cdots(\delta_n)$, where $\pi_{n+k} \in \Pi_{n+k}$, each k. Let ν_{n+k}^1 denote the marginal of ν for $H_{(n+k)}^{(n)}$, i.e., if $B \in \mathcal{B}(H_{(n+k)}^{(n)})$, then $\nu_{n+k}^1(B) = q_n\pi_{n+1}\cdots q_{n+k-1}(B|\delta_n)$. Let $\hat{\delta}_n$ be the element of $\mathcal{P}(H_{(n)} \times A_n)$ degenerate at δ_n, i.e., $\hat{\delta}_n(\{\delta_n\}) = 1$. If $B \in \mathcal{B}(H_{(n+k)})$, then $\hat{\delta}_n\nu_{n+1}^1(B) = \nu_{n+k}^1(B(\delta_n)) = \hat{\delta}_n\nu(B \times H^{(n+k)})$. In particular, by (4.4.6), $\hat{\delta}_n\nu_{n+k}^1(H_{n+k}^*) = \nu_{n+k}^1(H_{n+k}^*(\delta_n)) = 1$.

Let $\varepsilon > 0$ be given. By Lemma (4.4.10), for each k there exists π_{n+k}^* in Π_{n+k}^* and a closed subset $B_{n+k} \subseteq H_{n+k}^*$ such that $\pi_{n+k} = \pi_{n+k}^*$ on $B_{n+k} \cup H_{n+k}^{*\,c}$ and $\hat{\delta}_n\nu_{n+k}^1(B_{n+k}) > 1 - \varepsilon 2^{-k}$. Let $\nu^* = q_n\pi_{n+1}^*q_{n+1}\pi_{n+2}^*\cdots(\delta_n)$, and define $B = \cap_{k\in\mathbb{N}} B_{n+k} \times H^{(n+k)}$. Then $B \in \mathcal{B}(H)$, $\nu(B(\delta_n)) > 1 - \varepsilon$, and if

$B' \in \mathbb{B}(S_{n+1} \times H^{(n+1)})$ with $B' \subseteq B(\delta_n)$, then $\nu(B') = \nu^*(B')$. But then if $g: S_{n+1} \times H^{(n+1)}$ is bounded and continuous, then $|\int g d\nu - \int g d\nu^*| \leq 2\|g\| \nu(B(\delta_n)^c) < 2\|g\| \varepsilon$. Since ε was arbitrary and $\nu^* \in \mathscr{F}_n(\delta_n)$, it follows that every basic neighborhood [4: (5), p.236] of ν in $\mathscr{P}(S_{n+1} \times H^{(n+1)})$ contains an element of $\mathscr{F}_n^*(\delta_n)$, and hence $\nu \in \mathrm{cl}\, \mathscr{F}_n^*(\delta_n)$. $\square$

If $\pi^{n+1} \in \Pi^{*n+1}$, then $q\pi^{n+1}$ is a selection for $\mathscr{F}_n^*$ and hence also for $\mathscr{F}_n$. These selections are continuous on Δ_n.

Lemma (4.4.13). If $\pi^{n+1} \in \Pi^{*n+1}$, then $q\pi^{n+1}|\Delta_n$ is continuous.

Proof: Suppose $\pi^{n+1} = \{\pi_m\}_{m \geq n+1} \in \Pi^{*n+1}$. Since Δ_n and $\mathscr{P}(S_{n+1} \times H^{(n+1)})$ are metrizable, sequential convergence determines continuity. By an argument similar to [4: p.30], it suffices to show that the finite dimensional distributions of $q\pi^{n+1}$ are continuous on Δ_n.

This is true by assumption for q_n. To show it for $q_n\pi_{n+1}$, let $\hat{g}: S_{n+1} \times A_{n+1} \to \mathbb{R}$ be bounded and continuous, and let $g: \Delta_n \times S_{n+1} \to \mathbb{R}$ be given by

$$g(\delta_n, s_{n+1}) = \int_{A_{n+1}} \hat{g}(s_{n+1}, a_{n+1}) \pi_{n+1}(da_{n+1} | \delta_n, s_{n+1}).$$

Then g is bounded and measurable and by the usual mapping theorems [4: Theorems 5.4, 5.5], $g|_{H^*_{n+1}}$ is continuous since $\pi_{n+1}|_{H^*_{n+1}}$ is. Let $\{\delta_n^k\}_{k \in \mathbb{N}} \subseteq \Delta_n$ with $\lim_k \delta_n^k = \delta_n^0$, and for each $k \geq 0$, define $g_k(s_{n+1}) = g(\delta_n^k, s_{n+1})$. If $s_{n+1}^0 \in \mathrm{int}\, \sigma_n(\delta_n^0)$ and $\{s_{n+1}^k\}_{k \in \mathbb{N}} \subseteq S_{n+1}$ with $\lim_k s_{n+1}^k = s_{n+1}^0$, then $\lim_k g_k(s_{n+1}^k) = g_0(s_{n+1}^0)$. For by Lemma (4.4.4) and its proof, there is an open set $K \subseteq \mathrm{int}\, \sigma_n(\delta_n^0)$ with compact closure and an open set $V \subseteq \Delta_n$ such that

$(\delta_n^0, s_{n+1}^0) \in V \times K$, and if $\delta_n \in V$, then $K \subseteq \text{int}\, \sigma_n(\delta_n)$. Thus for k sufficiently large, $(\delta_n^k, s_{n+1}^k) \in V \times K \subseteq H_{n+1}^*$, and the claimed convergence follows. By the uniform boundedness of $\{g_k\}_{k \in \mathbb{N}}$ and [4: Theorems 5.4, 5.5], it follows that $\lim_k \int \hat{g}\, dq_n \pi_{n+1}(\delta_n^k) = \int \hat{g}\, dq_n \pi_{n+1}(\delta_n^0)$ and thus that $\lim_k q_n \pi_{n+1}(\delta_n^k) = q_n \pi_{n+1}(\delta_n^0)$. Hence, $q_n \pi_{n+1}$ is continuous on Δ_n.

By way of induction, suppose $q_n \pi_{n+1} \cdots q_{n+k-1} \pi_{n+k}$ is continuous on Δ_n for some $k > 1$. By essentially the same arguments as above, one can show that $q_n \pi_{n+1} \cdots \pi_{n+k} q_{n+k}$ is continuous on Δ_n, that $q_{n+k} \pi_{n+k+1}$ is continuous on Δ_{n+k}, and finally that $q_n \pi_{n+1} \cdots q_{n+k} \pi_{n+k+1}$ is continuous on Δ_n.□

Lower hemi-continuity of $\mathcal{F}_n$ is now easy.

<u>Theorem</u> (4.4. 14). $\mathcal{F}_n^*|_{\Delta_n}$ and $\mathcal{F}_n|_{\Delta_n}$ are l.h.c.

Proof: By [20: 4, p.28] and Lemma (4.4. 12), $\mathcal{F}_n$ is l.h.c. on Δ_n if and only if $\mathcal{F}_n^*$ is. To show this, suppose $\delta_n^0 \in \Delta_n$ and $\{\delta_n^k\}_{k \in \mathbb{N}} \subseteq \Delta_n$ with $\lim_k \delta_n^k = \delta_n^0$, and suppose $\nu \in \mathcal{F}_n^*(\delta_n^0)$, $\nu = q\pi^{n+1}(\delta_n^0)$ for some π^{n+1} in Π^{*n+1}. Then for each $k \geq 1$, $q\pi^{n+1}(\delta_n^k) \in \mathcal{F}_n^*(\delta_n^k)$, and by Lemma (4.4. 13), $\lim_k q\pi^{n+1}(\delta_n^k) = q\pi^{n+1}(\delta_n^0) = \nu$. By [20: Theorem 2, p.27], $\mathcal{F}_n^*$ is l.h.c. at δ_n^0.□

Theorems (4.4. 9) and (4.4. 14) complete the proof of Theorem (4.2.2).

5. Intertemporal Choice

5.1. Derived Utility and Demand

Definition (5.1.1). For each positive integer n, let

$u_n: H_{(n)} \times A_n \times \mathcal{P}(S_{n+1} \times H^{(n+1)}) \to \mathbb{R}$ be defined by

$$u_n(h_{(n)},a_n,\nu) = \int u(\mathrm{proj}_C(h_{(n)},a_n,s_{n+1},h^{(n+1)}))\nu(d(s_{n+1},h^{(n+1)})).$$

Then $u_n(h_{(n)},a_n,\nu)$ gives the agent's expected utility at date n when the future is governed by ν and experience to date is $(h_{(n)},a_n)$. It is clear from (A.2) that u_n is bounded and continuous in the product topology.

Definitions (5.1.2). The agent's derived utility function v_n at date n is simply the supremum of expected utility u_n over attainable futures, i.e., $v_n: H_{(n)} \times A_n \to \mathbb{R}$ is defined by

$$v_n(h_{(n)},a_n) = \sup\{u_n(h_{(n)},a_n,\nu): \nu \in \mathcal{F}_n(h_{(n)},a_n)\}.$$

Associated with this supremum is the set of measures that attain it:
$\zeta_n: H_{(n)} \times A_n \Rightarrow \mathcal{P}(S_{n+1} \times H^{(n+1)})$ given by

$$\zeta_n(h_{(n)},a_n) = \{\nu: \nu \in \mathcal{F}_n(h_{(n)},a_n),\ v_n(h_{(n)},a_n) = u_n(h_{(n)},a_n,\nu)\}.$$

Lemma (5.1.3). v_n is defined and bounded on $H_{(n)} \times A_n$, $v_n|_{\Delta_n}$ is continuous, and $\zeta_n|_{\Delta_n}$ is non-empty, compact valued and u.h.c.

Proof: Follows directly from Theorem (4.2.2) and [20: Corollary, p.30].□

Remarks (5.1.4). Since we are not interested in the values of v_n off Δ_n, but it would be convenient to have v_n bounded and continuous on all of $H_{(n)} \times A_n$, we simply extend $v_n|_{\Delta_n}$ to a bounded continuous function on all of $H_{(n)} \times A_n$.

By Lemma (3.2.1), the existence of such an extension is guaranteed. (c.f. [12: Theorem 5.1, p.149]). We fix one such extension and refer to it as v_n throughout the sequel.

The first stage of the two stage optimization process mentioned in Section 3.2 is depicted in (5.1.2). It consists of choosing a best attainable future at each date. The second stage consists of choosing a best current action that is feasible, where "best" refers to the derived utility function v_n.

Definition (5.1.5). For each n, let v_n^*: dom $D_n \to \mathbb{R}$ be defined by $v_n^*(h_{(n)}) = \sup\{v_n(h_{(n)},a_n): a_n \in D_n(h_{(n)})\}$. The function v_n^* is analogous to the optimal value function in dynamic programming. The agent's demand relation d_n: dom $D_n \Rightarrow A_n$ is then given by

$$d_n(h_{(n)}) = \{a_n: a_n \in D_n(h_{(n)}), v_n^*(h_{(n)}) = v_n(h_{(n)},a_n)\}.$$

Lemma (5.1.6). v_n^* is bounded on dom $D_{(n)}$, $v_n^*|_{H_n^*}$ is continuous, and $d_n|_{H_n^*}$ is non-empty, compact valued, and u.h.c.

Proof: Follows directly from Theorems (3.2.3) and (3.2.6), Lemmas (3.2.8) and (5.1.3), and [20: Corollary, p.30]. □

5.2. Optimal Plans

An agent's intertemporal choice behavior, as depicted in (5.1.2) and (5.1.5), is consistent if it is equivalent to following an optimal plan, i.e., a solution to the problem (4.1.4). Such a solution is constructed here from measurable selections for the relations ζ_n and d_n.

Lemma (5.2.1). There exists a transition probability $m_n: \Delta_n \rightarrow \mathcal{P}(S_{n+1} \times H^{(n+1)})$ such that (i) $m_n(\delta_n) \in \zeta_n(\delta_n)$; (ii) m_n is $\mathbb{B}(\Delta_n)/\mathbb{B}(\mathcal{P}(S_{n+1} \times H^{(n+1)}))$ - measurable; (iii) $m_n(\delta_n)(B \times H^{(n+1)}) = q_n(B|\delta_n)$, for every B in $\mathbb{B}(S_{n+1})$.

Proof: The existence of m_n satisfying (i) and (ii) follows from Lemma (5.1.3) and [13: Theorem 1]. By (ii) and Lemma (A.II), m_n is a transition probability. By (i), for each δ_n in Δ_n, there exists a date n+1 feasible plan π^{n+1} in Π^{n+1} such that $m_n(\delta_n) = q\pi^{n+1}(\delta_n)$, and (iii) follows.□

Remarks (5.2.2). In general, the π^{n+1} in the last sentence of the above proof may depend upon the δ_n in Δ_n. In this case, though m_n selects from ζ_n, m_n itself need not be the tail of an optimal plan. Even if m_n were of the form $m_n(\delta_n) = q\pi^{n+1}(\delta_n)$ for some π^{n+1} in Π^{n+1} and every δ_n in Δ_n, and this held for every n, there is no guarantee in general that π^{n+1} would be the tail of π^n for every n. This is the essence of the existence question.

Lemma (5.2.3). There exists a function $\pi_n^*: H_{(n)} \rightarrow A_n$ such that (i) $\pi_n^*(h_{(n)}) \in D_n(h_{(n)}), \forall\, h_{(n)} \in \text{dom } D_n$; (ii) $\pi_n^*(h_{(n)}) \in d_n(h_{(n)}), \forall\, h_{(n)} \in H_n^*$; (iii) π_n^* is $\mathbb{B}(H_{(n)})/\mathbb{B}(A_n)$-measurable. Moreover, for any $h_{(n)}^o$ in $H_{(n)}^*$ and a_n^o in $d_n(h_{(n)}^o)$, there exists a function $\pi_n^o: H_{(n)} \rightarrow A_n$ satisfying (i) - (iii) such that $\pi_n^o(h_{(n)}^o) = a_n^o$.

Proof: By Lemma (5.1.6) and [25: Corollary 1], there exists a function $\hat{\pi}_n: H_n^* \rightarrow A_n$ that is $\mathbb{B}(H_n^*)/\mathbb{B}(A_n)$-measurable and that selects from $d_n|_{H_n^*}$. Let $\pi_n^* \equiv \hat{\pi}_n$ on H_n^* and let $\pi_n^* \equiv 0$ (the zero vector in $\mathbb{R}^{2\ell}$) on the complement of $H_{(n)}^*$ (in $H_{(n)}$). Since $H_n^* \in \mathbb{B}(H_{(n)})$ and $0 \in D_n(h_{(n)})$ for all $h_{(n)}$ in dom D_n, π_n^* satisfies (i) - (iii). The second statement of the Lemma follows from the fact that π_n^* can be modified on any singleton subset of H_n^* in the desired manner, preserving (i) - (iii).□

Conventions (5.2.4). The need to modify selectors for d_n for particular histories and actions and still have a measurable selection stems from the fact that equilibrium actions must be capable of being chosen by agents following feasible strategies. We identify any function π_n^* satisfying (5.2.3) (i) - (iii) with the transition probability degenerate at the values of π_n^*. With this identification it is clear that $\pi_n^* \in \Pi_n$. For any sequence of such functions $\{\pi_n^*\}_{n\in\mathbb{N}}$, define $m_n^*: H_{(n)} \times A_n \to \mathcal{P}(S_{n+1} \times H^{(n+1)})$ by $m_n^*(h_{(n)},a_n) = q_n\pi_{n+1}^* q_{n+1}\pi_{n+2}^* \cdots (h_{(n)},a_n)$. Then m_n^* is a transition probability [2: Theorem 2.7.2] and hence a measurable selection for $\mathcal{F}_n$ (Lemma A.II). It will follow from the next theorem that m_n^* also selects from ζ_n on Δ_n. In proving this result it is convenient to have the selections m_n of Lemma (5.2.1) defined on all of $H_{(n)} \times A_n$. One easy way to do this is to let $m_n = m_n^*$ on the complement of Δ_n, where m_n^* is defined above. Henceforth, we assume the m_n of (5.2.1) are extended in this fashion.

Theorem (5.2.5). If $\{\pi_n^*\}_{n\in\mathbb{N}}$ is a sequence each component of which satisfies Lemma (5.2.3) (i) - (iii), then for each n in $\mathbb{N}$ and δ_n in Δ_n,

$$(5.2.6) \qquad u_n(\delta_n,m_n^*(\delta_n)) = v_n(\delta_n)$$

$$= \int_{H_{n+1}^*(\delta_n)} v_{n+1}^*(\delta_n,s_{n+1})q_n(ds_{n+1}|\delta_n)$$

$$= \int_{\Delta_{n+1}(\delta_n)} v_{n+1}(\delta_n,s_{n+1},a_{n+1})q_n\pi_{n+1}^*(d(s_{n+1},a_{n+1})|\delta_n).$$

Proof: Fix n and let δ_n be in Δ_n. By Lemma (5.1.6) the first integral in (5.2.6) is well defined. By definition of π_{n+1}^*, $v_{n+1}^*(\delta_n,\cdot) = v_{n+1}(\delta_n,\cdot,\pi_{n+1}^*(\delta_n,\cdot))$ on $H_{n+1}^*(\delta_n)$. The last equality in (5.2.6) then follows from (4.4.6) and (4.4.7)

To prove the first two equalities, let ν be in $\mathcal{F}_n(\delta_n)$, $\nu = q\pi^{n+1}(\delta_n)$, some $\pi^{n+1} = \{\pi_m\}_{m\geq n+1}$ in Π^{n+1}. It follows from (5.1.2) and (5.1.5) and from (4.4.6) and (4.4.7) that

$$u_n(\delta_n,\nu) \leq \int_{\Delta_{n+1}(\delta_n)} v_{n+1}(\delta_n,s_{n+1},a_{n+1})q_n\pi_{n+1}(d(s_{n+1},a_{n+1})|\delta_n),$$

and for every s_{n+1} in $H^*_{(n+1)}(\delta_n)$, that

$$\int_{\Delta_{n+1}(\delta_n,s_{n+1})} v_{n+1}(\delta_n,s_{n+1},a_{n+1})\pi_{n+1}(da_{n+1}|\delta_n,s_{n+1}) \leq v^*_{n+1}(\delta_n,s_{n+1}).$$

Integrating on both sides of this last inequality with respect to $q_n(\delta_n)$ and using the last equality of (5.2.6) gives that

$$u_n(\delta_n,\nu) \leq \int_{\Delta_{n+1}(\delta_n)} v_{n+1}(\delta_n,s_{n+1},a_{n+1})q_n\pi^*_{n+1}(d(s_{n+1},a_{n+1})|\delta_n).$$

Since ν was arbitrary and $m^*_n(\delta_n) \in \mathcal{F}_n(\delta_n)$, we conclude that

$$(*) \quad u_n(\delta_n,m^*_n(\delta_n)) \leq v_n(\delta_n) \leq \int_{\Delta_{n+1}(\delta_n)} v_{n+1}(\delta_n,s_{n+1},a_{n+1})q_n\pi^*_{n+1}(d(s_{n+1},a_{n+1})|\delta_n).$$

Let $\{m_{n+k}\}_{k\geq 0}$ be a sequence of functions satisfying Lemma (5.2.1) (extended as in (5.2.4)). Let $\lambda_n = m_n(\delta_n)$ and for each k in $\mathbb{N}$, let

$\lambda_{n+k} = q_n\pi^*_{n+1}\cdots q_{n+k}\pi^*_{n+k}m_{n+k}(\delta_n)$ (recall that m_{n+k} is a transition probability and use [2: Theorem 2.7.2]). Then $\lambda_{n+k} \to m^*_n(\delta_n)$ in $\mathcal{P}(S_{n+1} \times H^{(n+1)})$ since the fi dimensional distributions of λ_{n+k} converge to those of $m_n^*(\delta_n)$ (c.f. [4: p.30]). Also for each k = 0,1,..., define

$$w_{n+k} = \int u(\mathrm{proj}_{\zeta}(\delta_n, s_{n+1}, h^{(n+1)})\lambda_{n+k}(d(s_{n+1}, h^{(n+1)})).$$

Since u is bounded and continuous, $\lim_k w_{n+k} = u_n(\delta_n, m_n^*(\delta_n))$. Since $m_n(\delta_n) \in \zeta_n(\delta_n)$, $w_n = v_n(\delta_n)$. Similarly, for $k > 0$ and $t \in \Delta_{n+k}(\delta_n)$, $u_{n+k}(\delta_n, t, m_{n+k}(\delta_n, t)) = v_{n+k}(\delta_n, t)$, and consequently (4.4.7) implies that

$$w_{n+k} = \int_{\Delta_{n+k}(\delta_n)} v_{n+k}(\delta_n, t) q_n \pi_{n+1}^* \cdots q_{n+k-1} \pi_{n+k}^*(dt \mid \delta_n).$$

It follows from this representation and (*) that $w_n \le w_{n+1}$. Using this representacion and an argument similar to that leading to (*), one can show that $w_{n+k} \le w_{n+k+1}$ for every k = 0,1,..., and hence that

$$\begin{aligned} v_n(\delta_n) &= w_n \le w_{n+1} \\ &= \int_{\Delta_{n+1}(\delta_n)} v_{n+1}(\delta_n, s_{n+1}, a_{n+1}) q_n \pi_{n+1}^*(d(s_{n+1}, a_{n+1}) \mid \delta_n) \\ &\le \lim_k w_{n+k} \\ &= u_n(\delta_n, m_n^*(\delta_n)) \le v_n(\delta_n). \end{aligned}$$

Thus equality holds throughout, proving (5.2.6).□

Remarks (5.2.7). Equation (5.2.6) is an optimality type equation for the problem (4.1.4). The second equality in (5.2.6) produces a version of the optimality criterion for non-stationary dynamic programming [24: (5.1)]. Existence of optimal non-randomized plans follows easily from this result.

Theorem (5.2.8). If $\pi^* = \{\pi_n^*\}_{n\in\mathbb{N}}$ is any sequence of functions each component of which satisfies Lemma (5.2.3) (i) - (iii), then $\pi^* \in \Pi$ and if $\pi \in \Pi$ and $h_{(1)} \in H_1^*$, then

$$\int_{H^{(1)}} u(\mathrm{proj}_C(h^{(1)}))\pi q(dh^{(1)}|h_{(1)}) \leq \int_{H^{(1)}} u(\mathrm{proj}_C(h^{(1)}))\pi^* q(dh^{(1)}|h_{(1)}).$$

Proof: Apply (5.2.6) for $n = 1$.□

Remark (5.2.9). Any sequence $\pi^* = \{\pi_n^*\}_{n\in\mathbb{N}}$ as hypothesized in Theorem (5.2.8) solves the problem (4.1.4). The interpretation of the components of such a plan as demand functions derives from the optimality of this plan. Similarly, the interpretation of d_n as the agent's demand relation derives from this optimality property. The behavior of π_n^* off H_n^* is irrelevant so long as it selects from D_n on dom D_n since, as will be demonstrated in the next section (Theorem (5.3.5)), d_n is empty valued off Q_n^*.

5.3. Concavity and Monotonicity

The concavity and monotonicity of u assumed in (A.2) is preserved to some extent in the derived utility function v_n. The concavity result is due to Jordan [21: Proposition 2.33].

Lemma (5.3.1). If $h_{(n)} \in \text{dom } D_n$, then $v_n(h_{(n)},\cdot)$ is concave on $D_n(h_{(n)})$. If $h_{(n)} \in H^*_{(n)}$, then $d_n(h_{(n)})$ is convex.

Proof: Suppose $h_{(n)} \in \text{dom } D_n$ and $a^i_n \in D_n(h_{(n)})$, $i = 1,2$. Then $\delta^i_n = (h_{(n)}, a^i_n) \in \Delta_n$ and there exists $\nu^i \in \zeta_n(\delta^i_n)$, $i = 1,2$, (Lemma (5.1.3)). If $0 < \beta < 1$, let $\hat{\beta}_n$, $\hat{\nu}$, and ν be as in (4.3.1). For $\delta_n = \beta\delta^1_n + (1 - \beta)\delta^2_n$, $\nu \in \mathcal{J}_n(\delta_n)$ by Lemma (4.3.2). Thus

$$
\begin{aligned}
v_n(\delta_n) &\geq u_n(\delta_n,\nu) \\
&= \int_{S_{n+1} \times \hat{H}^{(n+1)}} u(\text{proj}_C(\delta_n, \hat{\beta}_n(s_{n+1}, \hat{h}^{(n+1)}))\hat{\nu}(d(s_{n+1}, \hat{h}^{(n+1)})) \\
&\geq \int \{\beta u(\text{proj}_C(\delta^1_n, s_{n+1}, a^1_{n+1}, \ldots)) \\
&\qquad + (1 - \beta)u(\text{proj}_C(\delta^2_n, s_{n+1}, a^2_{n+1}, \ldots))\}\hat{\nu}(d(s_{n+1}, \hat{h}^{(n+1)})) \\
&= \beta u_n(\delta^1_n, \nu^1) + (1 - \beta)u_n(\delta^2_n, \nu^2) \\
&= \beta v_n(\delta^1_n) + (1 - \beta)v_n(\delta^2_n).
\end{aligned}
$$

If $h_{(n)} \in H^*_n$, the convexity of $D_n(h_{(n)})$ and concavity of $v_n(h_{(n)},\cdot)$ imply that $d_n(h_{(n)})$ is convex.□

The monotonicity result is the analog here of [18: Lemma 3.4(iii)]. Though the result is standard, as in the case of Lemma (5.3.1), the context is not and a proof is required.

Lemma (5.3.2). If $h_{(n)} \in \operatorname{dom} D_n$ and $a_n^i \in D_n(h_{(n)})$, $i = 1,2$, and if $a_n^1 \le a_n^2$, then $v_n(h_{(n)}, a_n^1) < v_n(h_{(n)}, a_n^2)$.

Proof: Let $h_{(n)}, a_n^1, a_n^2$ be as hypothesized and let $\delta_n^i = (h_{(n)}, a_n^i)$, $i = 1,2$. By Lemma (4.3.3), $\mathcal{F}_n(\delta_n^1) \subseteq \mathcal{F}_n(\delta_n^2)$. Thus for $\nu^1 \in \zeta_n(\delta_n^1)$,

$$
\begin{aligned}
v_n(\delta_n^1) &= u_n(\delta_n^1, \nu^1) \\
&\le \int \ddot{u}(\operatorname{proj}_C(\delta_n^2, s_{n+1}, h^{(n+1)}))\nu^1(d(s_{n+1}, h^{(n+1)})) \\
&\le v_n(\delta_n^2).
\end{aligned}
$$

If $a_n^i = (c_n^i, f_n^i)$ and $c_n^1 \le c_n^2$, then the first inequality is strict. Thus $v_n(h_{(n)}, \cdot)$ is non-decreasing on $D_n(h_{(n)})$ and strictly increasing in c_n.

By Lemma (4.3.4), $v_{n+1}(\delta_n^1, s_{n+1}, a_{n+1}) \le v_{n+1}(\delta_n^2, s_{n+1}, a_{n+1})$, for every s_{n+1} in $H_{n+1}^*(\delta_n^1) = H_{n+1}^*(\delta_n^2)$ and every a_{n+1} in $D_{n+1}(\delta_n^1, s_{n+1}) \cap D_{n+1}(\delta_n^2, s_{n+1})$. Since $a_n^1 \le a_n^2$, it follows that if $s_{n+1} \in H_{n+1}^*(\delta_n^1)$, then $D_{n+1}(\delta_n^1, s_{n+1}) \subseteq D_{n+1}(\delta_n^2, s_{n+1})$ and $D_{n+1}(\delta_n^1, s_{n+1}) + (f_n^2 - f_n^1, 0) \subseteq D_{n+1}(\delta_n^2, s_{n+1})$. Assume that $f_n^1 \le f_n^2$, and let π_{n+1}^* satisfy Lemma (5.2.3) (i) - (iii). It follows from the strict monotonicity of $v_{n+1}(\delta_n, s_{n+1}, \cdot)$ in c_{n+1} and the results just derived that

$$
v_{n+1}(\delta_n^1, s_{n+1}, \pi_{n+1}^*(\delta_n^1, s_{n+1})) < v_{n+1}(\delta_n^2, s_{n+1}, \pi_{n+1}^*(\delta_n^2, s_{n+1})),
$$

for every s_{n+1} in $H_{n+1}^*(\delta_n^1)$. From this inequality and (5.2.6) we have that

$$v_n(\delta_n^1) = \int_{H^*_{n+1}(\delta_n^1)} v_{n+1}(\delta_n^1, s_{n+1}, \pi^*_{n+1}(\delta_n^1, s_{n+1}))q_n(ds_{n+1}|\delta_n^1)$$

$$< \int_{H^*_{n+1}(\delta_n^2)} v_{n+1}(\delta_n^2, s_{n+1}, \pi^*_{n+1}(\delta_n^2, s_{n+1}))q_n(ds_{n+1}|\delta_n^2)$$

$$= v_n(\delta_n^2). \square$$

Corollary (5.3.3). If $h_{(n)} \in H_n^*$, then $\xi_n(h_{(n)}) \cdot d_n(h_{(n)}) = r_n(h_{(n)})$.

Proof: Follows directly from Lemmas (5.1.6) and (5.3.2).□

The next result is the analog here of [18: Theorem 3.5] and will be needed in [29].

Lemma (5.3.4). Let $\{h^k_{(n)}\}_{k \in \mathbb{N}} \subseteq H_n^*$ with $h^k_{(n)} \to h^0_{(n)}$, some $h^0_{(n)}$ in $H_{(n)}$. If $a_n^k \in d_n(h^k_{(n)})$, for each k in $\mathbb{N}$, and $a_n^k \to a_n^0$, and if D_n is l.h.c. at $h^0_{(n)}$, then $a_n^0 \in d_n(h^0_{(n)})$.

Proof: Follows from Lemmas (3.2.1) amd (5.1.3) essentially as in [18: Theorem 3.5].□

Theorem (3.2.2) states that the feasible action relation is unbounded at any history $h_{(n)}$ in dom D_n for which $\xi_n(h_{(n)}) \notin Q_n(\xi_1(h_{(n)}), \ldots, \xi_{n-1}(h_{(n)}))$. This result, however, does not preclude $D_n(h_{(n)})$ from being unbounded only in directions in which $v_n(h_{(n)}, \cdot)$ decreases, and hence does not preclude $d_n(h_{(n)})$ being non-empty at boundary points $h_{(n)}$ of H_n^*. This possibility is cleared up in our final result which makes the analogy with Green's fundamental characterization [18: Theorem 2.1] complete.

<u>Theorem</u> (5.3.5). If $h_{(n)} \in \text{dom } D_n$ but $\xi_n(h_{(n)}) \notin Q_n(\xi_1(h_{(n)}),\ldots,\xi_{n-1}(h_{(n)}))$, then $v_n^*(h_{(n)}) > v_n(h_{(n)},a_n)$ for all a_n in $D_n(h_{(n)})$.

Proof: Let $h_{(n)}$ be as hypothesized. If $\xi_n^1(h_{(n)}) \not> 0$, the result follows from Lemma (5.3.2). Therefore assume $\xi_n^1(h_{(n)}) > 0$. As in the proof of Theorem (3.2.2), there exists a vector y in $\mathbb{R}^\ell$, $y \neq 0$, such that $\xi_n^2(h_{(n)})\cdot y = 0$ and $x\cdot y \geq (>)\ 0$ for all x in $\Gamma'(\text{int }\Gamma')$, where Γ' is defined in the proof of Theorem (3.2.2). Thus, in particular, if $s_{n+1} \in \sigma_n(h_{(n)},\cdot)(\text{int } \sigma_n(h_{(n)},\cdot))$, then $\xi_{n+1}^1(s_{n+1}) \in \Gamma'(\text{int }\Gamma')$ and $\xi_{n+1}^1(s_{n+1})\cdot y \geq (>)\ 0$.

Let $\hat{a}_n = (\hat{c}_n,\hat{f}_n)$ be in $D_n(h_{(n)})$ and suppose by way of contradiction that $v_n^*(h_{(n)}) = v_n(h_{(n)},\hat{a}_n)$. Let $\bar{a}_n = (\bar{c}_n,\bar{f}_n)$ be such that $\bar{c}_n = \hat{c}_n$ and $\bar{f}_n = \hat{f}_n + y$. It is easily verified that $\bar{a}_n \in D_n(h_{(n)})$. Let $\hat{\delta}_n = (h_{(n)},\hat{a}_n)$ and $\bar{\delta}_n = (h_{(n)},\bar{a}_n)$. Then $H_{n+1}^*(\hat{\delta}_n) = H_{n+1}^*(\bar{\delta}_n) = \text{int } \sigma_n(\hat{\delta}_n) = \text{int } \sigma_n(\bar{\delta}_n)$, and if $s_{n+1} \in H_{n+1}^*(\hat{\delta}_n)$ then $D_{n+1}(\hat{\delta}_n,s_{n+1}) \subseteq D_{n+1}(\bar{\delta}_n,s_{n+1})$. In particular, if $a_{n+1} \in D_{n+1}(\hat{\delta}_n,s_{n+1})$, then $s_{n+1}\cdot a_{n+1} \leq s_{n+1}\cdot(\hat{f}_n,\omega_{n+2}) < s_{n+1}\cdot(\bar{f}_n,\omega_{n+2})$. Since $\hat{c}_n = \bar{c}_n$, it follows from Lemma (4.3.4) that for every $a_{n+1} \in D_{n+1}(\hat{\delta}_n,s_{n+1})$, $v_{n+1}(\hat{\delta}_n,s_{n+1},a_{n+1}) = v_{n+1}(\bar{\delta}_n,s_{n+1},a_{n+1})$.

But by the above strict inequality in the budget constraint, Lemma (5.3.2) implies that

$$\begin{aligned} v_{n+1}^*(\hat{\delta}_n,s_{n+1}) &= \max\{v_{n+1}(\hat{\delta}_n,s_{n+1},a_{n+1}):\ a_{n+1} \in D_{n+1}(\hat{\delta}_n,s_{n+1})\} \\ &= \max\{v_{n+1}(\bar{\delta}_n,s_{n+1},a_{n+1}):\ a_{n+1} \in D_{n+1}(\hat{\delta}_n,s_{n+1})\} \\ &< \max\{v_{n+1}(\bar{\delta}_n,s_{n+1},a_{n+1}):\ a_{n+1} \in D_{n+1}(\bar{\delta}_n,s_{n+1})\} \\ &= v_{n+1}^*(\bar{\delta}_n,s_{n+1}). \end{aligned}$$

Integrating the extremes in this inequality and using (5.2.6) gives

$$
\begin{aligned}
v_n^*(h_{(n)}) = v_n(h_{(n)},\hat{a}_n) &= \int_{H_{n+1}^*(\hat{\delta}_n)} v_{n+1}^*(\hat{\delta}_n, s_{n+1})q(ds_{n+1}|\hat{\delta}_n) \\
&< \int_{H_{n+1}^*(\bar{\delta}_n)} v_{n+1}^*(\bar{\delta}_n, s_{n+1})q(ds_{n+1}|\bar{\delta}_n) \\
&= v_n(h_{(n)},\bar{a}_n),
\end{aligned}
$$

a contradiction.□

6. Conclusion

This concludes the first part of a competitive analysis of a sequential exchange model with spot trading and unconditional futures contracting. The consistency and continuity of agent choice have been established in the absence of institutional arrangements for handling bankruptcy. These results are employed in the second part of this analysis [29] concerned with the existence of temporary competitive equilibria.

APPENDIX

The results collected in this appendix are used in the text. They are variants of or similar to published results, but they require proof or annotated references.

Lemma (A.I). Let r and s be positive integers, $B \subseteq \mathbb{R}^r$, and $\phi: B \Rightarrow \mathbb{R}^s$ with $\emptyset \neq \text{dom}\,\phi \subseteq B$. If (i) the graph of ϕ is closed in $B \times \mathbb{R}^s$, (ii) ϕ is convex valued, and (iii) there exists y in $\mathbb{R}^s$ such that $y \in \phi(x)$ for each x in dom ϕ, then ϕ is u.h.c. at each x in dom ϕ where $\phi(x)$ is compact.

Proof: Let $\bar{x}$ be in dom ϕ and assume that $\phi(\bar{x})$ is compact. Let $\bar{y}$ be such that $(x,\bar{y}) \in$ graph ϕ, for all x in dom ϕ. By (i), if $y \in \mathbb{R}^s$ but $y \notin \phi(\bar{x})$, there exists sets $V \subseteq \text{dom}\,\phi$ and $U \subseteq \mathbb{R}^s$ such that V is open in dom ϕ and $\bar{x} \in V$, U is open in $\mathbb{R}^s$, $y \in U$, and $\phi(V) \subseteq U^c$ (the complement of U) [20: (b), p.24].

Let $\varepsilon > 0$ be given and let $D = \{y: y \in \mathbb{R}^s, d(y,\phi(\bar{x})) = \varepsilon\}$, where $d(y,\phi(\bar{x})) = \inf\{\|y - y'\| : y' \in \phi(\bar{x})\}$. Then D is compact and clearly $D \cap \phi(\bar{x}) = \emptyset$. Thus for each y in D there is a pair of open sets (V,U) with the properties mentioned above. Since the U cover D, there is a finite number of points $y_1,\ldots,y_k$ in D and corresponding pairs of open sets $(V_1,U_1),\ldots,(V_k,U_k)$ such that $D \subseteq \bigcup_{i=1}^k V_i \equiv U$. For $V \equiv \bigcap_{i=1}^k V_i$, $V \subseteq \text{dom}\,\phi$, V is open in dom ϕ, and $\bar{x} \in V$. Also U is open in $\mathbb{R}^s$ and $\phi(V) \subseteq U^c \subseteq D^c$.

Let $D^- = \{y: d(y,\phi(\bar{x})) < \varepsilon\}$ and let $D^+ = \{y: d(y,\phi(\bar{x})) > \varepsilon\}$. Then $D^c = D^- \cup D^+$, $\phi(\bar{x}) \subseteq D^-$, and if $x \in V$, then by (ii) either $\phi(x) \subseteq D^-$ or $\phi(x) \subseteq D^+$. But $\bar{y} \in \phi(\bar{x}) \subseteq D^-$ and $\bar{y} \in \phi(x)$ for all x in V. This implies that $\phi(x) \subseteq D^-$ for all x in V, and hence ϕ is u.h.c. at $\bar{x}$ [20: p.22].□

<u>Lemma</u> (A.II). Let $(X,\mathcal{B})$ be a measurable space and Y a metric space. If $\phi: X \to \mathcal{P}(Y)$ is $\mathcal{B}/\mathbb{B}(\mathcal{P}(Y))$ - measurable, then ϕ is a transition probability. Conversely, if ϕ is a transition probability from X to Y and $\mathcal{P}(Y)$ is second countable, then $\phi: X \to \mathcal{P}(Y)$ is $\mathbb{B}/\mathbb{B}(\mathcal{P}(Y))$- measurable.

Proof: The basic results along this line are [11: Theorems 2.1 and 3.1]. Indeed, according to [11: Theorem 2.1], if $\mathcal{W}$ is the σ-algebra of subsets of $\mathcal{P}(Y)$ generated by the mappings $\nu \to \nu(B)$, B in $\mathbb{B}(Y)$, then ϕ is $\mathbb{B}/\mathcal{W}$ - measurable if and only if ϕ is a transition probability. In [11: Theorem 3.1], the equality of $\mathcal{W}$ and $\mathbb{B}(\mathcal{P}(Y))$ is asserted for Y a compact metric space. Since Y is then separable, which is equivalent to $\mathcal{P}(Y)$ being second countable [4: p.239], this is a special case of the lemma. The coincidence of $\mathcal{W}$ with $\mathbb{B}(\mathcal{P}(Y))$ when $(Y,\mathbb{B}(Y))$ is a standard Borel space is established in [34: Lemma 6.1] but the proof uses only the fact that $\mathcal{P}(Y)$ has a countable base. □

<u>Lemma</u> (A.III). Let X be a topological space and Y a separable metric space. If ν is a regular probability measure on $\mathbb{B}(X)$ ([2: 7.3.1]) and if ϕ is a transition probability from X to Y, then for every $\varepsilon > 0$ there is a closed set $B \subseteq X$ with $\nu(B) > 1 - \varepsilon$ such that $\phi|_B : B \to \mathcal{P}(Y)$ is continuous.

Proof: This is a generalization of Lusin's theorem [2: 4.3.17(b)]. A proof when X is a metric space is given in [23: Lemma 5.4]. The general case of the lemma follows from Lemma (A.II) and [37: Theorem 3.1(b), p.35]. □

<u>Lemma</u> (A.IV). Let X and Y be metric spaces and let $\phi: X \Rightarrow Y$. Define $\gamma: X \Rightarrow \mathcal{P}(Y)$ by $\gamma(x) = \{\nu: \nu \in \mathcal{P}(Y), \mathrm{supp}(\nu) \subseteq \phi(x)\}$. Then dom ϕ = dom γ.

If $B \subseteq \text{dom } \phi$ and $\phi|_B$ is u.h.c. and compact valued, then $\gamma|_B$ is u.h.c. and compact valued. If Y is separable and complete and $\phi|_B$ is continuous and compact valued, then $\gamma|_B$ is continuous and compact valued. If Y is a Hilbert space and $\phi|_B$ is l.h.c. with closed convex values, the $\gamma|_B$ is l.h.c. with closed convex values.

Proof: The first conclusion is clear. The proof in [23 : Lemma 5.5] can be relativised to $B \subseteq \text{dom } \phi$ to obtain the second and third conclusions. If Y is a Hilbert space, the relation u_n defined in the proof of [23: Lemma 5.5] can be shown to be a continuous function if $\phi|_B$ has closed convex values. The remainder of this proof then goes through.□

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FOOTNOTES

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2. See the comments in [18: p. 1114 and Remark 4.5].

3. See the comments at the end of [40: Section 2].

4. [1: p. vii].

5. All spaces X in this paper for which the topology of weak convergence for $\mathcal{P}(X)$ is of interest are metrizable, and the appropriate references are [4, 30].

6. The extension of the model here to futures contracts with varying but bounded contract length appears feasible though complicated. Such an extension is of interest in developing a term structure of futures contracts but is beyond the scope of this analysis.

7. Throughout, when the range of integration is unspecified, it is understood to be the whole space in question, in this case C.

8. The notation "cl" and "int" denote closure and interior relative to the ambient factor space of H in which the set lies. In the case in (2.3.2) this space is $S_{n+1}(=\mathbb{R}^{2\ell})$, which is equivalent to taking the closure in Ω^2 since Ω^2 is closed in $\mathbb{R}^{2\ell}$. The relative interior of a convex set is denoted by "ri.

9. The well known and ongoing controversy over whether or not futures prices are or should be unbiased estimates of spot prices is at the heart of theories of speculation and hedging [7, 17]. There is also a growing literature on the so-called "martingale" or unbiasedness property of asset prices in the rational expectations setting that indicates that this property need not and in general will not hold in such equilibria [26].

10. C.F. Definition (4.4.1). The term "posterity" is borrowed from [38: p. 968].

TEMPORARY COMPETITIVE EQUILIBRIA IN A SEQUENCE OF SPOT AND FUTURES MARKETS[1]

By

David C. Nachman

TEMPORARY COMPETITIVE EQUILIBRIA IN A SEQUENCE OF SPOT AND FUTURES MARKETS[1]

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This is the second part of a competitive analysis of an exchange economy with a sequence of spot and (unconditional) futures markets begun in [8]. It is shown that commonness and compatibility assumptions regarding agents' opinions, along with the "consistency" assumptions of [8], imply a specific support structure of these opinions. This structure implies that for particular price-action histories at a given market date individual and aggregate demand for futures contracts are bounded below by resources at the subsequent date. Existence of a sequence of temporary equilibria and an equilibrium path for the economy follow in a routine fashion.

1. Introduction

In recent years, considerable attention has been focused on the equilibrium analysis of incomplete market economies [11] with particular emphasis on temporary equilibria in economies that evolve in time [4]. This paper is the second part of a competitive analysis of an exchange economy where markets are open at each of an infinite sequence of dates for spot trading and limited unconditional futures contracting, and where agents at each date are uncertain only about prices that will prevail at future dates. The model is an extension of a two period one developed by Green [5]. The problem concerns the viability of a purely competitive exchange mechanism in this context of sequential trading with no institutional arrangements for handling bankruptcy. The burden of avoiding bankruptcy falls on agents' probabilistic opinions (expectations) regarding future prices, and these opinions become the focal point of the analysis.

The first part of this analysis, carried out in [8], deals with the consistency and continuity of individual agent choice at each market date and over the infinite horizon. Under more or less standard assumptions on preferences and endowments it is shown that if an agent's opinions regarding prices are consistent in an appropriate sense, then the agent's choice behavior is consistent and demand is upper hemi-continuous, compact and convex valued for appropriate price-action histories and exhibits desired behavior on the boundary of this set of histories [8: Section 5]. In particular, an extension of Green's fundamental result [5: Theorem 2.1] is obtained that characterizes, for each date and history at that date, the current prices at which current demand is well defined [8: Theorems (3.2.2) and (5.3.5)].

The second part of this analysis, carried out in this paper, deals with the consistency in the aggregate sense, of individually optimal sequential choice decentralized through the price mechanism at each market date. The objective is to establish the existence of a sequence of temporary competitive equilibria and an equilibrium path for an economy populated by agents as described in [8].

The usual existence arguments require that aggregate demand exhibit appropriate behavior over approaches to the boundary of the set of admissible prices (c.f. [4: Section 3.1]). Complications arise in deducing this behavior in a perfectly competitive model of exchange with futues markets where there are no a priori bounds on the size of futures contracts.[2] Green [5] shows that one can avoid imposing such bounds, which institutionally augment the competitive mechanism, provided agents' opinions about prices exhibit some degree of commonness. The perfectly competitive mechanism is thus preserved and the class of environments in which it operates effectively (allocates resources) is more clearly delineated.

In this paper, two assumptions are made concerning agents' opinions that ensure the existence of certain common price forecasts.[3] The first is necessary for aggregate demand to be well defined and is analogous to [5: (4.1)]. The second assumption concerns common price forecasts, say at date n for date n + 1, whose futures price component is a common spot price forecast for date n + 2. This assumption can be viewed as a dynamic analog of Green's commonness assumption [5: (4.2)].

A third assumption is made on the aggregate of agent's opinions that has no counterpart in [5] where existence of equilibrium for markets beyond the initial date is not considered. In the present model, the set of histories where aggregate demand for date n, n > 1, is well behaved (upper hemi-continuous) may be smaller than the set of admissible prices, where aggregate demand is well defined. A compatibility assumption on agents' opinions is made to obtain a set suitable for application of a market equilibrium theorem.

These three assumptions, together with the consistency assumptions of [8], imply a specific support structure of agents' opinions. As a consequence of this structure, at any date and any candidate for temporary equilibrium at that date each agent's demand for futures contracts on current markets is bounded below by the agent's known endowment at the subsequent market date. Thus at such candidate prices, agents choose contracts on current markets that they can honor at the next market date independently of prices at that date and thereby avoid bankruptcy.[4] Aggregate demand for futures contracts is bounded below by the total resources of the economy at the next market date. The existence of a temporary equilibrium for the economy at any date and any history of temporary equilibria up to that date follows in a routine fashion.

The three assumptions mentioned above are presented and discussed in Section 3. The result concerning the support structure of agents' opinions and the consequence of this result for individual demand for futures contracts are also presented in this section. In Section 4, equilibrium concepts are defined and the existence results are established. In Section 5, some concluding remarks are made on the restrictive nature of the basic assumptions and the support structure of opinions they entail.

For convenience, the basic structure, assumptions, and results of the model in [8] are reviewed in Section 2. The reader is referred to [8] for complete notation, terminology, definitions, etc., especially [8: Section 2].

2. The Basic Model

2.1 Markets and Agents

The setting is an exchange economy where markets are open at each of an infinite sequence of dates, indexed by the set $\mathbb{N}$ of positive integers, for spot trading and one-period unconditional futures contracting in $\ell \geq 2$ elementary (non-dated, non-storable) commodities. Terms of exchange at date n are expressed by a price system s_n, where $s_n \in S_n = \mathbb{R}^{2\ell}$, the price space for date n markets.[5]

The consumption set for an agent at date n is $C_n = \mathbb{R}^{\ell}_{+}$, and the set of admissible (net) futures contracts is $F_n = \mathbb{R}^{\ell}$.[6] Then $A_n = C_n \times F_n$ is the action space at date n, and $H_{(n)} = S_1 \times A_1 \times \cdots \times A_{n-1} \times S_n$ (with $H_{(1)} = S_1$) is the set of price-action histories through prices at date n. Similarly, $S_{(n)} = S_1 \times \cdots \times S_n$ is the set of price histories through date n, and $H^{(n)} = A_n \times S_{n+1} \times \cdots$ is the set of price-action histories from actions at n onward. The set of histories is $H = H_{(n)} \times H^{(n)}$, for any n, and $C = C_1 \times C_2 \times \cdots$ is the consumption set of an agent.

For any product space having $S_n(A_n)(C_n)(F_n)$ as a factor, $\xi_n(\xi_n^1)(\xi_n^2)(\alpha_n)(\alpha_n^1)(\alpha_n^2)$ will denote the projection of the product space onto S_n (the first ℓ coordinates of S_n) (the second ℓ coordinates of S_n) $(A_n)(C_n)(F_n)$. For example, if $s_n \in S_n$, then $\xi_n^1(s_n)$ is the spot price component of s_n and $\xi_n^2(s_n)$ is the futures price component of s_n. On the other hand, if $s_{(n)} \in S_{(n)}$, then $H_{(n)}[s_{(n)}] = \{h_{(n)}: h_{(n)} \in H_{(n)}, s_{(n)} = (\xi_1(h_{(n)}),\ldots,\xi_n(h_{(n)}))\}$, the set of price-action histories with common price history $s_{(n)}$.

The typical agent is identified by a 4-tuple $< b_0, \omega, \precsim, q >$. The vector b_0 is an element of $\mathbb{R}^{\ell}$ and represents the preexisting contracts of the agent at date 1. If $b_{0j} < (>) \ 0$, the agent has contracted to deliver (receive) $|b_{0j}|$ units of commodity j at date 1. The sequence $\omega = \{\omega_n\}_{n \in \mathbb{N}}$ is a known naturally occurring endowment with ω_n the endowment vector ($\omega_n \in \mathbb{R}^{\ell}$) received at date n. The relation $\precsim$ is a preference ordering on the space $\mathcal{P}(C)$ of probability measures on (the Borel sets of) C.[7] An agent's opinion is represented by the sequence $q = \{q_n\}_{n \in \mathbb{N}}$, where q_n: $H_{(n)} \times A_n \rightarrow \mathcal{P}(S_{n+1})$ is a transition probability. If $(h_{(n)}, a_n)$ is the history of prices and actions for the agent up through date n, then $q_n(h_{(n)}, a_n)$ is the agent's subjective probability measure over date n + 1 (equilibrium) price systems.

2.2 Assumptions

The assumptions made in [8] on agents' characteristics are listed here for the sake of reference. The reader is referred to [8: Section 2.3] for details and discussion of these assumptions. The first two concern endowments and preferences.

Assumption (A.1): $\omega_1 + b_0 \in \mathbb{R}^{\ell}_{+0}$, and $\omega_n \in \mathbb{R}^{\ell}_{++}$, $n \geq 2$.

Assumption (A.2): There exists a bounded continuous function u: $C \rightarrow \mathbb{R}$ such that (i) if $\nu^1, \nu^2 \in \mathcal{P}(C)$, $\nu^1 \precsim \nu^2$ if and only if $\int u d\nu^1 \leq \int u d\nu^2$; (ii) u is concave on C; and (iii) if $c^1, c^2 \in C$ and $c^1 \leq c^2$, then $u(c^1) < u(c^2)$.

To state the assumptions on agents' opinions, some further notation is required. For each n and history $(h_{(n)}, a_n)$, let $\sigma_n(h_{(n)}, a_n) = \text{supp}(q_n(h_{(n)}, a_n))$, the support of the probability measure $q_n(h_{(n)}, a_n)$. The correspondence σ_n has closed values, but they may be unbounded. Under Assumption (A.3) below, it will suffice for technical purposes to deal with a compact base for the closure of the convex cone generated by σ_n. To this end, if $x = (x_1,\ldots,x_m) \in \mathbb{R}^m$, $x \neq 0$, let $\tilde{x} = x(\Sigma_{j=1}^m |x_j|)^{-1}$, and if $B \subseteq \mathbb{R}^m$, let $\tilde{B} = \{\tilde{x}: x \in B,\ x \neq 0\}$. The desired relation [8] is $\bar{\sigma}_n: H_{(n)} \times A_n \Rightarrow \Omega^2$ defined by $\bar{\sigma}_n(h_{(n)},a_n) = \mathrm{cl}(\tilde{\sigma}_n(h_{(n)},a_n))$, where Ω^2 is the unit simplex in $\mathbb{R}^{2\ell}$, and "cl" denotes closure.[9]

Henceforth, unless specifically stated otherwise, the quantifier "for each n in $\mathbb{N}$" is understood in definitions, assumptions, theorems, etc. Also, if the range of a variable is left unspecified in such statements it is understood to be the relevant projection or factor space of H. For example, in (A.3)(i) to follow, the range of $(h_{(n)},a_n)$ and $(h'_{(n)},a'_n)$ is $H_{(n)} \times A_n$.

Assumption (A.3): (i) If $\xi_k(h_{(n)},a_n) = \xi_k(h'_{(n)},a'_n)$, $k = 1,\ldots,n$, then $q_n(h_{(n)},a_n) = q_n(h'_{(n)},a'_n)$; (ii) q_n is continuous and the values of σ_n are convex subsets of $\mathbb{R}_+^{2\ell}$; (iii) $\bar{\sigma}_n$ is u.h.c. on $H_{(n)} \times A_n$; (iv) $q_n(\mathrm{int}\ \sigma_n(h_{(n)},a_n)|h_{(n)},a_n) = 1$.

Interpretations and immediate implications of the various parts of Assumption (A.3) are given in [8: Remarks (2.3.4) - (2.3.8)]. In light of (A.3)(i), q_n, σ_n and objects derived from them are alternately written as functions of the entire history or as functions defined on $S_{(n)}$ as convenience dictates. In light of the continuity assumption (A.3)(iv) that the boundary of $\sigma_n(\cdot)$ has subjective probability zero, the price systems in int $\sigma_n(\cdot)$ are referred to as price forecasts (at the relevant history). For a price forecast s_{n+1}, $\tilde{s}_{n+1}(\xi^1_{n+1}(s_{n+1}))$ $(\xi^2_{n+1}(s_{n+1}))$ is a relative (spot) (futures) price forecast [8: Definition (2.3.9)].

Consistency and continuity of agent choice depends significantly on the set of price systems Q_1 and the relation Q_n, $n \geq 2$, defined formally in [8: Definition (2.3.10)]. The set Q_1 consists of price systems s, for date 1 whose futures price component $\xi_1^2(s_1)$, perhaps after a scale change, is a spot price forecast for date 2 <u>at the history</u> s_1, i.e., for some $\lambda > 0$ and $p \in \mathbb{R}^\ell$, $(\lambda\xi_1^2(s_1),p) \in \text{int } \sigma_1(s_1)$. Similarly, for $n \geq 2$ and price history $s_{(n-1)}$ through date n-1, $Q_n(s_{(n-1)})$ is the set of price systems s_n for date n markets whose futures price component $\xi_n^2(s_n)$, up to a scale change, is a spot price forecast for date n+1 <u>at the history</u> $(s_{(n-1)},s_n)$. The importance of these prices follows from [8: Theorem (3.2.2)]. According to this result, if an agent's set of feasible actions at date n and history $h_{(n)}$ is non-empty, it is bounded if and only if $\xi_n(h_{(n)}) \in Q_n(\xi_1(h_{(n)}),\ldots,\xi_{n-1}(h_{(n)}))$ $(= Q_1$ if $n = 1)$. In light of this result, the following assumption is made.

<u>Assumption</u> (A.4): $Q_1 \neq \emptyset$ and for $n \geq 2$ and $s_{(n-1)} \in S_{(n-1)}$, $\text{int } \sigma_{n-1}(s_{(n-1)}) \subseteq Q_n(s_{(n-1)})$.

The inclusion assumed in (A.4) can be interpreted as a weak form of an expectations hypothesis. It states that the futures component of every price forecast for date n is, up to a scale change, a forecast of spot prices for date n+1, where the term "forecast" is used in the sense described earlier as opposed to the stronger sense of conditional mean forecast. See [8: Remarks (2.3.12), (2.3.13)] for comments and problems regarding such unbiased forecasts and the corresponding expectations hypothesis.

The graph of the correspondence int σ_n, considered as a subset of $H_{(n+1)}$, is denoted by $\sum_n^o$. Under (A.3), $\sum_n^o$ is open in $H_{(n+1)}$ [8: Remark (2.3.8)]. Q_1 is open in S_1, and for $n \geq 2$, Q_n is open valued and has an open graph in $S_{(n)}$ [8: Remark (2.3.11)]. For convenience, opinions satisfying (A.3) and (A.4) are referred to simply as consistent.

2.3 Demand Relations

Informally, an action for an agent at date n consists of a vector c_n of elementary commodities for consumption and a vector $b_n = (b_{n1},..,b_{n\ell})$ of futures contracts, where $b_{nj} > (<) 0$ indicates a contract to receive (deliver) $|b_{nj}|$ units of commodity j at date n+1. The convention $f_{n-1} = \omega_n + b_{n-1}$, for all $n \in \mathbb{N}$, defines net futures contracts made at date n-1, and an action at date n is formally defined to be a vector $a_n = (c_n, f_n) \in A_n$ [$\underset{\sim}{8}$: (3.1.1)].

Solvency and feasibility of action at any date are determined relative to the price system prevailing at that date and the agent's known endowment augmented by futures contracts made at the previous date. For a given (agent specific) history $h_{(n)} \in H_{(n)}$, $(\alpha^2_{n-1}(h_{(n)}), \omega_{n+1})$ denotes that part of the agent's known endowment relevant for markets at date n augmented by total futures contracts $b_{n-1} = \alpha^2_{n-1}(h_{(n)}) - \omega_n$ made at date n-1, and $r_n(h_{(n)}) = \xi_n(h_{(n)}) \cdot (\alpha^2_{n-1}(h_{(n)}), \omega_{n+1})$ is the present value of this augmented endowment at prevailing date n prices $\xi_n(h_{(n)})$. If $r_n(h_{(n)}) \geq 0$, the agent is defined to be solvent at $h_{(n)}$, while if $r_n(h_{(n)}) < 0$, the agent is defined to be bankrupt at $h_{(n)}$ [$\underset{\sim}{8}$: (3.1.2)].

Action at date n and history $h_{(n)}$ is feasible if the agent is solvent at $h_{(n)}$ [$\underset{\sim}{8}$: (3.1.2)] and agents plan to be solvent with subjective certainty [$\underset{\sim}{8}$: (3.1.4)]. The feasible action action relation for an agent, denoted by D_n, is defined in [$\underset{\sim}{8}$: (3.1.6)]. At histories $h_{(n)}$ for which $r_n(h_{(n)}) \geq 0$, $D_n(h_{(n)})$ consists of those actions a_n that satisfy the budget constraint $\xi_n(h_{(n)}) \cdot a_n \leq r_n(h_{(n)})$ and the planning constraint that $r_{n+1}(h_{(n)}, a_n, s_{n+1}) \geq 0$ for all $s_{n+1} \in \sigma_n(h_{(n)}, a_n)$ [$\underset{\sim}{8}$: Remark (3.1.5)]. If $r_n(h_{(n)}) < 0$, then $D_n(h_{(n)})$ is empty, by definition [$\underset{\sim}{8}$: (3.1.2), (3.1.3)]. The domain of D_n is denoted by dom D_n($= \{h_{(n)}: D_n(h_{(n)}) \neq \emptyset\}$), and its graph by Δ_n.

With regard to consistency and continuity of agent choice, the following sets of histories are the important ones. Let $H_1^* = Q_1$ and for $n \geq 2$, let $H_n^* = (\Delta_{n-1} \times S_n) \cap \Sigma_{n-1}^o$. Under assumptions (A.1) - (A.4), $D_n | H_n^*$ (the restriction of D_n to H_n^*) is non-empty, compact and convex valued, and continuous [8: Section 3.2]. An agent's infinite horizon expected utility maximization problem is cast in the framework of dynamic programming in [8: Section 4.1] and consistency and continuity of intertemporal choice in this context is established in [8: Section 5]. In particular, an agent's date n demand relation, denoted by $d_n : \mathrm{dom}\, D_n \Rightarrow A_n$, is defined in [8: (5.1.5)]. This relation is non-empty, compact, and convex valued and u.h.c. on H_n^* [8: Lemmas (5.1.6),(5.3.1)], satisfies the budget constraint with equality on H_n^* [8: Corollary (5.3.3)], and is such that $d_n(h_{(n)}) = \emptyset$ if $\xi_n(h_{(n)}) \notin Q_n(\xi_1(h_{(n)}), \ldots, \xi_{n-1}(h_{(n)}))$ [8: Theorem (5.3.5)].

3. Opinions in the Aggregate

The economy consists of a finite number of agents indexed by the set $\mathcal{I} = \{1,..,I\}$, where $I \in \mathbb{N}$. If $i \in \mathcal{I}$, all agent i specific objects are denoted with a presuperscript i. Thus agent i's characteristics are written $< {}^i b_0, {}^i\omega, {}^i\precsim, {}^i q >$, agent i's date n demand relation is ${}^i d_n$, ${}^i H_1^* = {}^i Q_1$, and ${}^i H_n^* = ({}^i\Delta_{n-1} \times S_n) \cap {}^i\Sigma^o_{n-1}$, $n \geq 2$, etc. The economy is $E = \{< {}^i b_0, {}^i\omega, {}^i\precsim, {}^i q > : i \in \mathcal{I}\}$. Assumptions (A.1) - (A.4) are maintained for each $i \in \mathcal{I}$ throughout the sequel.

3.1. Common Price Forecasts

Definition (3.1.1): For each $n \geq 2$, let $P_n : S_{(n-1)} \Rightarrow S_n$ be defined by $P_n(s_{(n-1)}) = \cap_{i\in\mathcal{I}} \text{int } {}^i\sigma_{n-1}(s_{(n-1)})$. Then $P_n(s_{(n-1)})$ is the set of price forecasts for date n common to all agents at the history $s_{(n-1)}$. Define the sets $\mathbb{P}_n$ inductively as follows: Let $\mathbb{P}_1 = \cap_{i\in\mathcal{I}} {}^i Q_1$, and for $n \geq 2$, let $\mathbb{P}_n = \{s_{(n)} = (s_{(n-1)}, s_n) \in S_{(n)} : s_{(n-1)} \in \mathbb{P}_{n-1}, s_n \in P_n(s_{(n-1)})\}$.

The only candidates for a temporary competitive equilibrium for date 1 markets are those price systems, if any, in $\mathbb{P}_1$ since $\mathbb{P}_1 = \cap_{i\in\mathcal{I}} {}^i H_1^*$ is the domain of aggregate demand for date 1. This follows from [8: Theorem (5.3.5)]. The importance of common price forecasts for the equilibrium existence question for date n, $n \geq 2$, is similar. For a given price history $s_{(n-1)}$, if $i \in \mathcal{I}$ and $({}^i h_{(n-1)}, {}^i a_{n-1}) \in H_{n-1}[s_{(n-1)}] \times A_{n-1} \cap {}^i\Delta_{n-1}$, then ${}^i d_n({}^i h_{(n-1)}, {}^i a_{n-1}, \cdot)$ is non-empty, compact, and convex valued and u.h.c. on $P_n(s_{(n-1)})$. Thus the candidates for a finite sequence of temporary equilibria up through date n must include those price histories, if any, in the set $\mathbb{P}_n$. In this regard, the following assumption seems reasonable.

Assumption (A.5): $\mathbb{P}_1 \neq \emptyset$ and if $n \geq 2$ and $s_{(n-1)} \in \mathbb{P}_{n-1}$, then $P_n(s_{(n-1)}) \neq \emptyset$.

Remark (3.1.2): In words, (A.5) ensures the existence of histories of common price forecasts beginning from prices in the domain of aggregate demand for date 1. The assumption that $\mathbb{P}_1 \neq \emptyset$ is analogous to [5: (4.1)]. Note, however, that for $n \geq 2$, $P_n(s_{(n-1)})$ is not necessarily the set of admissible prices in the usual sense, i.e., the domain of aggregate demand, and hence the non-emptyness of $P_n(s_{(n-1)})$ is not strictly necessary for the existence of a temporary equilibrium at date n (see the comments below preceeding (A.7)). The behavior of aggregate demand off $P_n(s_{(n-1)})$ is not well understood though, and (A.5) is made in the face of this ignorance.

Remark (3.1.3): The last part of (A.5) is implied by a "common expectations" hypothesis under the first part of (A.5), (A.3)(i), the convexity assumption of (A.3)(ii), (A.3)(iv), and an integrability assumption. To formulate this hypothesis, let ${}^iq = \{{}^iq_n\}_{n \in \mathbb{N}}$, $i \in \mathcal{A}$, be opinions satisfying these assumptions. Using notation similar to that in [8: Remark (2.3.12)], let ${}^iE(\xi_{n+1}|\xi_1,..,\xi_n)(s_{(n)})$ denote suggestively the vector integral $\int s_{n+1}\, {}^iq_n(ds_{n+1}|s_{(n)})$, which is assumed to be finite (in each corrdinate) for all i and $s_{(n)}$. Expectations are said to be common at n and $s_{(n)}$ if ${}^iE(\xi_{n+1}|\xi_1,..,\xi_n)(s_{(n)}) = {}^{i'}E(\xi_{n+1}|\xi_1,..,\xi_n)(s_{(n)})$ for all i, i', and in this case the common expectation is written without presuperscript. The "common expectations" hypothesis is that expectations are common at n on $\mathbb{P}_n$, for each n. By assumption $\mathbb{P}_1 \neq \emptyset$. By convexity in (A.3)(ii) and by (A.3)(iv), ${}^iE(\xi_{n+1}|\xi_1,..,\xi_n)(s_{(n)}) \in \operatorname{int} {}^i\sigma_n(s_{(n)})$ for each i and $s_{(n)}$. The last part of (A.5) follows from the hypothesis by repeated application of this result, which says simply that (common) expectations are (common) price forecasts.

It is clear that the essence of the commonness assumption in [$\underset{\sim}{5}$: (4.2)] is not that there exists common price forecasts,[11] but rather that there are common price forecasts that are invariant with respect to the history at which the forecast is made. What this invariance implies can be seen in the context of the present model for $n = 1$. Under (A.4), for every s_1 in $\mathbb{P}_1$, $\xi_1^2(s_1)$ is, up to agent specific changes in scale, a common spot price forecast for date 2 <u>at the history</u> s_1, i.e., for each i there exists $\lambda_i > 0$ and p^i in $\mathbb{R}^\ell$ such that $(\lambda_i \xi_1^2(s_1), p^i) \in \text{int } {}^i\sigma_1(s_1)$. If $s_1 \in \text{cl } \mathbb{P}_1$, but $s_1 \notin \mathbb{P}_1$, this is no longer the case since s_1 lies outside of ${}^iQ_1$ for one or more i. As illustrated in the proof in [$\underset{\sim}{5}$: Theorem 4.11], Green's assumption applied here would guarantee the existence of $\hat{s}_1$ in $\mathbb{P}_1$ such that $\xi_1^2(\hat{s}_1)$ is a common spot price forecast in the above sense at <u>every</u> history s_1.[12]

The following assumption can be viewed as a dynamic anolog of this result on common spot price forecasts. For $B \subseteq \mathbb{R}^m$, $B \neq \emptyset$, let $\Gamma(B)$ denote the smallest cone containing B.

<u>Assumption</u> (A.6): If $n \in \mathbb{N}$ and $s_{(n+1)} = (s_{(n)}, s_{n+1}) \in \mathbb{P}_n \times \text{cl } P_{n+1}(s_{(n)})$, then there exists $\hat{s}_{n+1}$ in $P_{n+1}(s_{(n)})$ such that $\xi_{n+1}^2(\hat{s}_{n+1}) \in \bigcap_{i \in \mathcal{A}} \xi_{n+2}^1(\Gamma(\text{int } {}^i\sigma_{n+1}(s_{(n+1)})))$.

<u>Remark</u> (3.1.4): Under (A.4), the consequence in (A.6) holds automatically for all histories of common price forecasts, i.e., for all $s_{(n+1)}$ in $\mathbb{P}_{n+1}$. Take $\hat{s}_{n+1} = \xi_{n+1}(s_{(n+1)})$. This may no longer be automatic for a history $s_{(n+1)} = (s_{(n)}, s_{n+1})$ if $s_{(n)} \in \mathbb{P}_n$ but $s_{n+1} \in \partial P_{n+1}(s_{(n)})$ (the boundary of $P_{n+1}(s_{(n)})$) since s_{n+1} may lie outside ${}^iQ_{n+1}(s_{(n)})$ for one or more i. This will perforce be the case under (A.7) below for all $s_{(n)} \in \mathbb{P}_n$ and some $s_{n+1} \in \partial P_{n+1}(s_{(n)})$. The essence of (A.6) then is that for such price histories there exists a common price

forecast $\hat{s}_{n+1}$ at $s_{(n)}$ such that $\xi^2_{n+1}(\hat{s}_{n+1})$ is a common spot price forecast for date n +2 at the history $(s_{(n)}, s_{n+1})$.[13]

Remark (3.1.5): Note that unlike the invariance involved in the implication of [5: (4.2)], the $\hat{s}_{n+1}$ of (A.6) may depend on the specific history $(s_{(n)}, s_{n+1})$. The invariance involved in [5: (4.2)] can be achieved under the common expectations hypothesis of (3.1.3) and the additional hypothesis that expectations also produce spot price forecasts in the sense that for each i and $(s_{(n)}, s_{n+1})$, $\xi^2_{n+1}({}^{i}E(\xi_{n+1}|\xi_1,..,\xi_n)(s_{(n)})) \in \xi^1_{n+2}(\Gamma(\text{int } {}^{i}\sigma_{n+1}(s_{(n)}, s_{n+1})))$. If expectations are common at n on $\mathbb{P}_n$ and $s_{(n)} \in \mathbb{P}_n$, then $\hat{s}_{n+1} = E(\xi_{n+1}|\xi_1,..,\xi_n)(s_{(n)})$ satisfies the consequence in (A.6) for every s_{n+1}. Under reasonable assumptions, e.g., those listed in (3.1.3), an agent's conditional expectation ${}^{i}E(\xi_{n+1}|\xi_1,..,\xi_n)(s_{(n)})$ is a price forecast for n+1 at $s_{(n)}$. The additional hypothesis says that the futures component of this price forecast is a spot price forecast for n+2 at $s_{(n)}$ and any price system s_{n+1} at date n+1.[14]

The final assumption on agents' opinions involves obtaining a set of prices suitable for application of a market equilibrium theorem, e.g., the theorem of Debreu [3]. For $n = 1$ there is no problem so long as $\mathbb{P}_1$ is convex (c.f. [5: Remark 4.8]). For $n > 1$ and appropriate personal histories with common price history $s_{(n-1)}$, aggregate demand on date n markets will be non-empty, compact, and convex valued and u.h.c. on the set $P_n(s_{(n-1)})$ of common price forecasts at $s_{(n-1)}$. For date n prices outside the larger (under (A.4)) set $\bigcap_{i \in \mathcal{A}} {}^{i}Q_n(s_{(n-1)})$, demand for some agent, and hence aggregate demand, will not be defined [8: Theorem (5.3.5)]. The following example indicates that this set is indeed larger in general and may not be suitable for an existence argument.

Example (3.1.6): Let $B \subseteq \mathbb{R}^{\ell}_{++}$, B compact and convex, with int $B \neq \emptyset$. Let ${}^{i}\sigma_n \equiv B \times B$ and let ${}^{i}q_n$ be identically normalized Lebesgue measure on ${}^{i}\sigma_n$, for each i and n. Clearly (A.3) holds, and for each i and n, int ${}^{i}\sigma_n \equiv$ int $B \times$ int $B \subseteq {}^{i}Q_n \equiv \mathbb{R}^{\ell}_{++} \times \Gamma(\text{int } B)$. Thus (A.4) holds and $\mathbb{P}_1 = \mathbb{R}^{\ell}_{++} \times \Gamma(\text{int } B)$, $P_n \equiv$ int $B \times$ int B, $n > 1$, verifying (A.5) and (A.6). But for $n > 1$ and personal histories $({}^{i}h_{(n-1)}, {}^{i}a_{n-1})$ with ${}^{i}h_{(n-1)} \in H_{(n-1)}[s_{(n-1)}]$, some $s_{(n-1)}$, and ${}^{i}a_{n-1} \in {}^{i}d_{n-1}({}^{i}h_{(n-1)})$, there exists prices $s_n \in \mathbb{R}^{\ell}_{++} \times \Gamma(\text{int } B)$ such that some agent is bankrupt at s_n unless $\alpha^2_{n-1}({}^{i}a_{n-1}) \in \mathbb{R}^{\ell}_{+}$ for each i, i.e., unless every agent's net demand for futures contracts at date n-1 is non-negative.

While non-negative net demand for futures contracts makes the existence question much easier, this result is difficult to guarantee without further assumptions on agent's preferences or opinions. The problem in the above example is that $P_n(s_{(n-1)})$ and $\bigcap_{i \in \mathcal{A}} {}^{i}Q_n(s_{(n-1)})$ do not generate the same cone of prices. This is ruled out explicitly in the following assumption. For $B \subseteq \mathbb{R}^m$, $B \neq \emptyset$ and convex, B is said to have a base for $\Gamma(B)$ if there is a convex subset $B' \subseteq B$ such that if $x \in \Gamma(B)$, $x \neq 0$, then x has a unique representation of the form $x = \lambda b$, some scalar $\lambda > 0$ and some b in B' [10: 3.1.,p.25].

Assumption (A.7): $\mathbb{P}_1$ is convex and for $n \geq 2$ and $s_{(n-1)} \in \mathbb{P}_{n-1}$, $\Gamma(P_n(s_{(n-1)})) = \Gamma(\bigcap_{i \in \mathcal{A}} {}^{i}Q_n(s_{(n-1)}))$, and $P_n(s_{(n-1)})$ has a relatively compact base for $\Gamma(P_n(s_{(n-1)}))$.

Remark (3.1.7): The assumption that $\mathbb{P}_1$ is convex is analogous to [5: (3.4)]. The assumption that $P_n(s_{(n-1)})$ and $\bigcap_{i \in \mathcal{A}} {}^{i}Q_n(s_{(n-1)})$ generate the same cone ensures that along any approach to the boundary of this cone from within $P_n(s_{(n-1)})$, if a limit exists and the limit point is not in $P_n(s_{(n-1)})$, it will not be in ${}^{i}Q_n(s_{(n-1)})$ for some i. The assumption that $P_n(s_{(n-1)})$ has a relatively compact base for this cone ensures that one can get to the boundary of the cone from within $P_n(s_{(n-1)})$.

3.2 The Support Structure of Opinions

The size of futures contracts demanded by an agent at any date is constrained by the intertemporal consistency requirement that the agent plan to be solvent at the next date with subjective certainty. The implied constraint on choice at date n is expressed formally by the correspondence ${}^{i}D_n^2$ defined in [8: (3.1.4)]. This constraint can be stated in terms of the support correspondence ${}^{i}\sigma_n$ as follows. If $h_{(n)} \in H_{(n)}[s_{(n)}]$, then $a_n = (c_n, f_n) \in {}^{i}D_n^2(h_{(n)})$ if and only if $c_n \geqq 0$ and ${}^{i}r_{n+1}(h_{(n)}, a_n, s_{n+1}) = s_{n+1} \cdot (f_n, {}^{i}\omega_{n+2}) \geq 0$ for every $s_{n+1} \in {}^{i}\sigma_n(s_{(n)})$. Equivalently, and perhaps more conveniently, $a_n = (c_n, f_n) \in {}^{i}D_n^2(h_{(n)})$ if and only if $c_n \geqq 0$ and $\bar{s}_{n+1} \cdot (f_n, {}^{i}\omega_{n+2}) \geq 0$ for every $\bar{s}_{n+1} \in {}^{i}\bar{\sigma}_n(s_{(n)})$ [8: Lemma (3.2.5)].

Since ${}^{i}\omega_{n+2}$ is positive, by (A.1), how negative the coordinates of f_n can be depends significantly on the agent's relative price forecasts or their limits $\bar{s}_{n+1}$. To see the extreme case, for $j = 1, \ldots, 2\ell$, let $e_j = (e_{1j}, \ldots, e_{2\ell j})$, where $e_{kj} = 0$ if $k \neq j$, and $e_{jj} = 1$. Suppose that for some j, $1 \leq j \leq \ell$, $e_j \in {}^{i}\bar{\sigma}_n(s_{(n)})$. Then $e_j \cdot (f_n, {}^{i}\omega_{n+2}) = f_{nj}$ and hence $(c_n, f_n) \in {}^{i}D_n^2(h_{(n)})$ only if $f_{nj} \geq 0$. Similarly, if $\{e_j : j = 1, \ldots, \ell\} \subseteq {}^{i}\bar{\sigma}_n(s_{(n)})$, the extreme case, then for $c_n \geqq 0$, $(c_n, f_n) \in {}^{i}D_n^2(h_{(n)})$ if and only if $f_n \geqq 0$, i.e., ${}^{i}D_n^2(h_{(n)}) = \mathbb{R}_+^{\ell} \times \mathbb{R}_+^{\ell}$.

In words, in the extreme case, demand for net futures contracts will be nonnegative, and hence total futures contracts demanded $b_n = f_n - {}^{i}\omega_{n+1}$ will be bounded below by the agent's resources at the next date, n + 1. Let Ω^1 denote the unit simplex in $\mathbb{R}^{\ell}$.

Definition (3.2.1): For each i and n, define ${}^i\mathcal{B}_n = \{s_{(n)} \in S_{(n)}: \Omega^1 \times \{0\} \subseteq {}^i\bar{\sigma}_n(s_{(n)})\}$. Since ${}^i\bar{\sigma}_n(s_{(n)})$ is convex by (A.3)(ii), the extreme case obtains at $s_{(n)}$ for agent i if and only if $s_{(n)} \in {}^i\mathcal{B}_n$.

The principal result of this work is that the extreme case is common at common price forecast histories and their limits.

Theorem (3.2.2): Under (A.3) - (A.7), $\mathrm{cl}\, \mathbb{P}_n \subseteq {}^i\mathcal{B}_n$ for each $i \in \mathcal{L}$ and $n \in \mathbb{N}$.

The proof of this result is given in Section 3.3. The basic consequence of the extreme case is stated formally in the following.

Corollary (3.2.3): If $s_{(n)} \in \mathrm{cl}\, \mathbb{P}_n$, then for each $i \in \mathcal{L}$ and ${}^ih_{(n)} \in H_{(n)}[s_{(n)}]$, ${}^iD_n^2({}^ih_{(n)}) = \mathbb{R}_+^{2\ell}$.

3.3 Proof of Theorem (3.2.2)

Let $n \in \mathbb{N}$ be given and let $s_{(n)} \in \mathbb{P}_n$, which is non-empty by (A.5). The objective is to show that $\Omega^1 \times \{0\} \subseteq {}^i\bar{\sigma}_n(s_{(n)})$ for each i. The same conclusion will then follow for $s_{(n)}$ in $\mathrm{cl}\, \mathbb{P}_n$ since ${}^i\bar{\sigma}_n$ has a closed graph by (A.3)(iii). The proof is by contradiction using a fixed point argument. Since the point $s_{(n)}$ remains fixed throughout the proof, objects such as $P_{n+1}(s_{(n)})$ and ${}^i\bar{\sigma}_n(s_{(n)})$ are abbreviated to P_{n+1} and ${}^i\bar{\sigma}_n$, respectively. For clarity, the restriction of the mapping $x \to \tilde{x}$ to $\mathbb{R}_{+0}^{2\ell}$ will be denoted by ρ, i.e., $\rho(x) = x(\Sigma_{j=1}^{2\ell} x_j)^{-1}$, if $x \in \mathbb{R}_{+0}^{2\ell}$, but where convenient, images under ρ will be denoted by a tilde. Thus $\rho(P_{n+1})$ is abbreviated to $\tilde{P}_{n+1}$. Clearly ρ is continuous and maps open (convex) sets to open (convex) sets. By (A.3)(ii) and (A.5) it follows that $\tilde{P}_{n+1}$ is a non-empty open convex subset of $\mathrm{ri}\,\Omega^2$.

The basic outline of the proof is as follows. For each i, $\mathrm{cl}\, \tilde{P}_{n+1} \subseteq {}^i\bar{\sigma}_n$, and it suffices to show that $\{e_j : j=1, \ldots, \ell\} \subseteq \mathrm{cl}\tilde{P}_{n+1}$. Define $\tilde{P}^*_{n+1} = \mathrm{cl}\tilde{P}_{n+1} \cap \tilde{P}^c_{n+1} \cap \mathbb{R}^{2\ell}_{++}$. Suppose that for some j, $1 \leq j \leq \ell$, $e_j \notin \mathrm{cl}\tilde{P}_{n+1}$.[15] Under this assumption it is shown that there exists a point x^* in $\tilde{P}^*_{n+1}$ such that for some scalar $\lambda > 0$, $\lambda x^* \in \bigcap_{i \in \mathcal{L}} {}^iQ_{n+1}(s_{(n)})$. It follows then by (A.7) that $x^* \in \Gamma(P_{n+1})$ and hence that $x^* \in \tilde{P}_{n+1}$, which is the contradiction. The argument is broken into several steps.

Step 1. The set $\tilde{P}^*_{n+1}$ is not necessarily convex, but an appropriate image of it is. Let $\Omega^2_j = \{x \in \Omega^2 : x_j = 0\}$ and let $\beta : \Omega^2 \setminus \{e_j\} \to \Omega^2_j$ be given by $\beta(x) = e_j + (1 - x_j)^{-1}(x - e_j)$. Then $\beta(x)$ is the unique point in Ω^2_j on the line through the closed segment $[e_j, x]$.

Lemma (3.3.1): β is continuous and maps open (convex) sets to open (convex) sets.

Proof: Continuity of β is clear from its specification. Let $B \subseteq \Omega^2 \setminus \{e_j\}$ be open, let $\hat{x} \in \beta(B)$, and let $x \in B$ such that $\hat{x} = \beta(x)$. Let $\varepsilon > 0$ be such that $B_\varepsilon(x) \cap \Omega^2 \setminus \{e_j\} \subseteq B$, where $B_\varepsilon(x) = \{x' \in \mathbb{R}^{2\ell} : \| x - x' \| < \varepsilon\}$. If $\hat{x}' \in B_\varepsilon(\hat{x}) \cap \Omega^2_j$, let $x' = x_j e_j + (1 - x_j)\hat{x}'$. Since $0 \leq x_j < 1$, $x' \in \Omega^2_j \setminus \{e_j\}$ and $\beta(x') = \hat{x}'$. Also, $\|x - x'\| = (1 - x_j) \|\hat{x} - \hat{x}'\| < \varepsilon$, and hence $x' \in B$. It follows that $B_\varepsilon(\hat{x}) \cap \Omega^2_j \subseteq \beta(B)$ and thus that $\beta(B)$ is open in Ω^2_j. If B is convex and $\hat{x}^k \in \beta(B)$, $x^k \in B$, with $\hat{x}^k = \beta(x^k)$, $k = 1, 2$, for any λ, $0 < \lambda < 1$, define $\gamma = \lambda(1 - x^2_j)(\lambda(1 - x^2_j) + (1 - \lambda)(1 - x^1_j))^{-1}$. Then $0 < \gamma < 1$ and $\lambda\hat{x}^1 + (1 - \lambda)\hat{x}^2 = \beta(\gamma x' + (1 - \gamma)x^2)$, and hence $\beta(B)$ is convex.□

By Lemma (3.3.1), the set $\beta(\mathrm{cl}\tilde{P}_{n+1})$ is a compact convex subset of Ω^2_j. This set will serve as the domain and range of a map to which a fixed-point theorem will be applied. To simplify notation one step further, the image $\beta(\mathrm{cl}\tilde{P}_{n+1})$ will be denoted by $\bar{\beta}_{n+1}$ and the image $\beta(\tilde{P}_{n+1})$ by β_{n+1}.

Step 2. There are several steps in constructing the appropriate mapping. The first is to map $\overline{\beta}_{n+1}$ back into $\mathrm{cl}\tilde{P}_{n+1}$. Fox $\hat{x}$ in β_{n+1}, let $x \in \tilde{P}_{n+1}$ be such that $\hat{x} = \beta(x)$ and define $\lambda(x) = \sup\{\lambda \in (0,1) : \lambda e_j + (1-\lambda)x \in \tilde{P}_{n+1}\}$. Then $0 < \lambda(x) < 1$ since $e_j \notin \mathrm{cl}\tilde{P}_{n+1}$ and since $x \in \tilde{P}_{n+1} = \mathrm{ri}\ \tilde{P}_{n+1}$ [12: Theorem 6.4]. Define $g : \beta_{n+1} \to \mathrm{cl}\tilde{P}_{n+1}$ by letting $g(\hat{x}) = \lambda(x)e_j + (1-\lambda(x))x$, for any $x \in \tilde{P}_{n+1}$ with $\hat{x} = \beta(x)$. By definition of $\lambda(x)$, $g(\hat{x})$ does not depend on the particular x and $g(\hat{x}) \in \mathrm{cl}\tilde{P}_{n+1} \cap \tilde{P}_{n+1}^{c}$ [9: Corollary 2, p. 21]. Since $P_{n+1} \subseteq \mathbb{R}^{2\ell}_{++}$, $g(\hat{x}) \in \mathbb{R}^{2\ell}_{++}$ and thus $g : \beta_{n+1} \to \tilde{P}^{*}_{n+1}$. In particular $\tilde{P}^{*}_{n+1} \neq \emptyset$. To complete this step, define the correspondence $\hat{g} : \overline{\beta}_{n+1} \to \mathrm{cl}\tilde{P}_{n+1}$ by $\hat{g}(\hat{x}) = \{g(\hat{x})\}$ if $\hat{x} \in \beta_{n+1}$, and $\hat{g}(\hat{x}) = [e_j, \hat{x}] \cap \mathrm{cl}\tilde{P}_{n+1}$ if $\hat{x} \in \overline{\beta}_{n+1} \cap \beta^{c}_{n+1}$, where $[e_j, \hat{x}]$ is the closed line segment joining e_j and $\hat{x}$.

Lemma (3.3.2): g is continuous on β_{n+1} and $\hat{g}$ is compact valued and u.h.c. on $\overline{\beta}_{n+1}$.

Proof: Let $\hat{x} \in \beta_{n+1}$ and let $\{\hat{x}^k\}_{k \in N} \subseteq \beta_{n+1}$ with $\lim_k \hat{x}^k = \hat{x}$. Let $\{x, x^k\}_{k \in N} \subseteq \tilde{P}_{n+1}$ with $\hat{x} = \beta(x)$ and $\hat{x}^k = \beta(x^k)$. The sequence $\{x^k\}$ can be chosen so that $\lim_k x_k = x$, essentially as in the proof of Lemma (3.3.1), by taking a decreasing sequence of balls about x and $\hat{x}$. Let $\{(\lambda(x^{k'}), g(\hat{x}^{k'}))\}$ be a subsequence of $\{(\lambda(x^k), g(\hat{x}^k))\}$ that converges to some point (λ^0, x^0) in $[0,1] \times \mathrm{cl}\tilde{P}_{n+1}$. Then $x^0 = \lambda^0 e_j + (1-\lambda^0)x$, $\lambda^0 < 1$, and hence $x^0 \in \mathbb{R}^{2\ell}_{++}$. Also, $x^0 \notin \tilde{P}_{n+1}$ since $g(\hat{x}^k) \in \tilde{P}^{*}_{n+1} \subseteq \tilde{P}^{c}_{n+1}$, which is closed in Ω^2. Thus $x^0 \in \tilde{P}^{*}_{n+1}$ and by Lemma (3.3.1), $\beta(x^0) = \lim_k \beta(g(\hat{x}^{k'})) = \beta(x) = \hat{x}$. By the uniqueness of $g(\hat{x})$, $x^0 = g(\hat{x})$, proving that g is continuous at $\hat{x}$.

Clearly, $\mathrm{dom}\ \hat{g} = \overline{\beta}_{n+1}$ and $\hat{g}$ is compact and convex valued on $\mathrm{dom}\ \hat{g}$. By Lemma (3.3.1), β_{n+1} is open in Ω^2_j and thus $\hat{g}$ is u.h.c. on β_{n+1}. Suppose $\hat{x} \in \overline{\beta}_{n+1}$ and let $\{\hat{x}^k\} \subseteq \overline{\beta}_{n+1}$ with $\lim_k \hat{x}^k = \hat{x}$. For each k, let $x^k \in \hat{g}(\hat{x}^k)$. Since $\mathrm{cl}\tilde{P}_{n+1}$ is compact, there exists a subsequence $\{x^{k'}\}$ of $\{x^k\}$ converging to some $x^0 \in \mathrm{cl}\tilde{P}_{n+1}$. By definition of $\hat{g}$, for each k there exists $\lambda^k \in [0,1)$ such that $x^k = \lambda^k e_j + (1 - \lambda^k)\hat{x}^k$. It follows that $\lambda^{k'}$ converges and hence that $x^0 \in [e_j, \hat{x}] \cap \mathrm{cl}\tilde{P}_{n+1} = \hat{g}(\hat{x})$. By [6: Theorem 1, p.24], $\hat{g}$ is u.h.c. at $\hat{x}$. □

Step 3. The next step is to scale things up from $\tilde{P}_{n+1}$ to P_{n+1}. By (A.7), P_{n+1} has a relatively compact base P'_{n+1} for the cone $\Gamma(P_{n+1})$. Let $P^*_{n+1} = \mathrm{cl}P_{n+1} \cap (P_{n+1})^c \cap \mathbb{R}^{2\ell}_{++}$.

Lemma (3.3.3): $\mathrm{cl}\tilde{P}_{n+1} = \rho(\mathrm{cl}P_{n+1})$ and $\tilde{P}^*_{n+1} \subseteq \rho(P^*_{n+1})$.

Proof: Since ρ is continuous, $\rho(\mathrm{cl}P_{n+1}) \subseteq \mathrm{cl}\tilde{P}_{n+1}$. To go the other way, suppose $x \in \mathrm{cl}\tilde{P}_{n+1}$ and let $\{x^k\} \subseteq \tilde{P}_{n+1}$ with $\lim_k x^k = x$. Let $\{y^k\} \subseteq P'_{n+1}$ with $x^k = \rho(y^k)$, and let $\{y^{k'}\}$ be a subsequence of $\{y^k\}$ converging to some y in $\mathrm{cl}P'_{n+1}$. By [10: Proposition 3.2, p. 25], $y \neq 0$, and hence $\lim_{k'} \rho(y^{k'}) = \rho(y)$. Thus $x = \rho(y) \in \rho(\mathrm{cl}P_{n+1})$, and the equality is proved. If in addition $x \in \tilde{P}^*_{n+1}$, then $y \notin P_{n+1}$ but $y \in \mathbb{R}^{2\ell}_{++}$ since $x \in \mathbb{R}^{2\ell}_{++}$. Consequently, $y \in P^*_{n+1}$, proving the second inclusion. □

Lemma (3.3.4): $\mathrm{cl}P'_{n+1}$ is a compact base for $\Gamma(\mathrm{cl}P_{n+1})$ and ρ is a homeomorphism between $\mathrm{cl}P'_{n+1}$ and $\mathrm{cl}\tilde{P}'_{n+1}$.

Proof: By [10: Proposition 3.6, p.26], there is a strictly positive (with regard to the order in $\mathbb{R}^{2\ell}$ determined by the cone $\Gamma(P_{n+1}) \cup \{0\}$) linear functional L on $\mathbb{R}^{2\ell}$ such that $P'_{n+1} = L^{-1}(1) \cap \Gamma(P_{n+1})$. Since L is then continuous, $L^{-1}(1)$ is closed and hence $\mathrm{cl}P'_{n+1} \subseteq L^{-1}(1) \cap \Gamma(\mathrm{cl}P_{n+1})$. Suppose $y \in L^{-1}(1) \cap \Gamma(\mathrm{cl}P_{n+1})$. Then for some $\lambda > 0$, $\lambda y \in \mathrm{cl}P_{n+1}$ and there exists a sequence $\{\bar{y}^k\} \subseteq P_{n+1}$ such that $\lim_k \bar{y}_k = \lambda y$. For each k, there exists $y^k \in P'_{n+1}$ and $\lambda^k > 0$ such that $\bar{y}^k = \lambda^k y^k$, uniquely. It follows that $\lambda^{-1}\lambda^k y^k \to y$. By passing to a subsequence if necessary, it may be assumed that λ^k converges to some $\lambda' > 0$ and that y^k converges to some $y' \in \mathrm{cl}P'_{n+1}$. Then

$\lambda^{-1}\lambda'y' = y$ and $y,y' \in L^{-1}(1)$ imply that $\lambda^{-1}\lambda' = 1$ and hence $y = y'$. Therefore $clP'_{n+1} = L^{-1}(1) \cap \Gamma(clP_{n+1})$. If $y \in \Gamma(clP_{n+1})$, $y \neq 0$, it also follows by this same argument that $L(y) = L(\lambda^{-1}\lambda'y') = \lambda^{-1}\lambda' > 0$, and hence L is strictly positive with regard to the order in $\mathbb{R}^{2\ell}$ determined by the cone $\Gamma(clP_{n+1}) \cup \{0\}$. By [10: Proposition 3.6, p. 26], clP'_{n+1} is a base for $\Gamma(clP_{n+1})$.

It follows from the properties of such a base and Lemma (3.3.3) that ρ restricted to clP'_{n+1} is one-to-one and onto $cl\tilde{P}_{n+1}$. Since ρ is continuous and clP'_{n+1} is compact, ρ so restricted is a homeomorphism. □

Henceforth, let ρ_1 denote ρ restricted to clP'_{n+1} and let ρ_2 denote the inverse of ρ_1.

<u>Step</u> 4. A crucial step in this construction relates common price forecasts and their limits to other common price forecasts as suggested in (A.6). For each $i \in \mathcal{L}$, define the relation ${}^i\psi : clP_{n+1} \Rightarrow P_{n+1}$ by ${}^i\psi(y) =$ $\{y' \in P_{n+1} : \xi^2_{n+1}(y') \in \xi^1_{n+2}(\Gamma(\text{int } {}^i\sigma_{n+1}(s_{(n)},y)))\}$. Similarly, define the aggregate relation $\psi : clP_{n+1} \Rightarrow P_{n+1}$ by $\psi(y) = \cap_{i \in \mathcal{L}} {}^i\psi(y)$.

<u>Lemma</u> (3.3.5): dom ${}^i\psi$ = dom $\psi = clP_{n+1}$, and ${}^i\psi$ and ψ are open and convex valued and l.h.c. on clP_{n+1}, each $i \in \mathcal{L}$.

Proof: That ${}^i\psi$ is open and convex valued and l.h.c. on dom ${}^i\psi$ follows from (A.3)(ii). If dom $\psi = clP_{n+1}$, then ψ inherits these properties of ${}^i\psi$ [6: 6(2), p.35]. That dom $\psi = clP_{n+1}$ is precisely what (A.6) states. □

<u>Step</u> 5. Since ψ takes values in P_{n+1}, its image under ρ takes values in $\tilde{P}_{n+1}$, but a simple mapping of ψ into $\tilde{P}_{n+1}$ would loose the identity of appropriate values of this correspondence. This identity can be preserved by using ρ_2 first. Define the correspondence $\tilde{\psi} : cl\tilde{P}_{n+1} \Rightarrow \tilde{P}_{n+1}$ by $\tilde{\psi}(x) = \rho(\psi(\rho_2(x)))$.

<u>Lemma</u> (3.3.6): $\tilde{\psi}$ is open and convex valued and l.h.c. on dom $\tilde{\psi} = \mathrm{cl}\tilde{P}_{n+1}$.

Proof: If $x \in \mathrm{cl}\tilde{P}_{n+1}$, then $\rho_2(x) \in \mathrm{cl}P'_{n+1} \subseteq \mathrm{cl}P_{n+1}$. The result then follows from Lemmas (3.3.4) and (3.3.5). □

<u>Step</u> 6. It follows from Lemma (3.3.6) and [7: Theorem 3.10‴(c)] that $\tilde{\psi}$ admits a continuous selection, i.e., there exists a continuous function $f : \mathrm{cl}\tilde{P}_{n+1} \to \mathbb{R}^{2\ell}$ such that for each $x \in \mathrm{cl}\tilde{P}_{n+1}$, $f(x) \in \tilde{\psi}(x)$. This function is the final part needed for the construction. For each $\hat{x} \in \overline{\beta}_{n+1}$, let $\hat{f}(\hat{x}) = \beta(\mathrm{co}(f(\hat{g}(\hat{x}))))$, where $f(\hat{g}(\hat{x}))$ is the image under f of the set $\hat{g}(\hat{x})$ and $\hat{f}(\hat{x})$ is the image under β of the convex hull of this set.

<u>Lemma</u> (3.3.7): dom $\hat{f} = \overline{\beta}_{n+1}$, $\hat{f}$ is compact and convex valued and u.h.c. on $\overline{\beta}_{n+1}$, and $\hat{f}(\hat{x}) \subseteq \beta_{n+1}$ for each $\hat{x} \in \overline{\beta}_{n+1}$.

Proof: By Lemmas (3.3.1), (3.3.2), continuity of f, and [1: Theorem 1', p. 113], $\hat{f}$ is compact and convex valued and u.h.c. on dom $\hat{f}$. By Lemma (3.3.6), dom $\hat{f} = \overline{\beta}_{n+1}$. By construction, $f(\hat{g}(\hat{x})) \subseteq \tilde{\psi}(\hat{g}(\hat{x})) \subseteq \tilde{P}_{n+1}$, and by convexity of $\tilde{P}_{n+1}$, $\hat{f}(\hat{x}) \subseteq \beta(\tilde{P}_{n+1}) \equiv \beta_{n+1}$. □

<u>Step</u> 7. By Kakutani's theorem [1: p. 174], the correspondence $\hat{f}$ has a fixed-point, i.e., there exists $\hat{x}^* \in \overline{\beta}_{n+1}$ such that $\hat{x}^* \in \hat{f}(\hat{x}^*)$. By Lemma (3.3.7), $\hat{x}^* \in \beta_{n+1}$ and thus $\hat{g}(\hat{x}^*) = \{g(\hat{x}^*)\}$ and $\hat{f}(\hat{x}^*) = \{\beta(f(g(\hat{x}^*)))\}$, implying that $\hat{x}^* = \beta(f(g(\hat{x}^*)))$. Let $x^* = g(\hat{x}^*)$. It remains to verify that x^* is the vector sought at the outset.

By construction of g, $x^* \in \tilde{P}^*_{n+1}$. By definition of β, $\hat{x}^* = \beta(f(x^*)) = e_j + (1 - f_j(x^*))^{-1}(f(x^*) - e_j)$, where $f_j(x^*)$ is the $j^{\underline{th}}$ coordinate of $f(x^*)$. It follows that $\xi^2_{n+1}(\hat{x}^*) = (1 - f_j(x^*))^{-1}\xi^2_{n+1}(f(x^*))$. Also by construction of g, $x^* = g(\hat{x}^*)$ implies that $\hat{x}^* = \beta(x^*)$, and hence that $\xi^2_{n+1}(\hat{x}^*) = (1 - x^*_j)^{-1}\xi^2_{n+1}(x^*)$. It follows that $\xi^2_{n+1}(f(x^*)) = (1 - f_j(x^*))(1 - x^*_j)^{-1}\xi^2_{n+1}(x^*)$.

Let $\gamma_1 = (1 - f_j(x^*))(1 - x_j^*)^{-1} > 0$, and let $y^* = \rho_2(x^*)$. Then for some $\gamma_2 > 0$, $\gamma_2 y^* = x^*$. Since f selects from $\tilde{\psi}$, $f(x^*) \in \tilde{\psi}(x^*) = \rho(\psi(y^*))$, and hence for some $\gamma_3 > 0$, $\gamma_3 f(x^*) \in \psi(y^*)$. By definition of ψ and these results, $\gamma_1\gamma_2\gamma_3\xi^2_{n+1}(y^*) = \gamma_1\gamma_3\xi^2_{n+1}(x^*) = \gamma_3\xi^2_{n+1}(f(x^*))$ and $\gamma_3\xi^2_{n+1}(f(x^*)) \in \xi^1_{n+2}(\Gamma(\text{int}\ {}^i\sigma_{n+1}(s_{(n)}, y^*)))$, for each $i \in \mathcal{A}$. Thus $\gamma_2^{-1}x^* = y^* \in \bigcap_{i\in\mathcal{A}} {}^iQ_{n+1}(s_{(n)})$, and the proof is complete.

4. Equilibrium

4.1 The Initial Condition

Throughout Section 4, the economy E is assumed to satisfy (A.1) - (A.7). For each n, $\bar{\omega}_n = \Sigma_{i \in \mathcal{I}}\, {}^i\omega_n$ denotes the total resources of E available at date n. In terms of these resources, (A.1) suffices for the study of individual agent behavior and for aggregate behavior for n > 1. For aggregate behavior at date 1, the following initial condition is needed.

Assumption (A.0): $\Sigma_{i \in \mathcal{I}} ({}^i\omega_1 + {}^ib_0) = \bar{\omega}_1 \in \mathbb{R}^{\ell}_{++}$.

The equality in (A.0) states that the aggregate of preexisting contracts is consistent with (a preexisting) equilibrium in futures contracts, i.e., $\Sigma_{i \in \mathcal{I}}\, {}^ib_0 = 0$. The second condition in (A.0) states that every commodity is in positive supply at the first market date. Assumption (A.0) is also maintained throughout Section 4.

4.2 Temporary Competitive Equilibrium

Notation (4.2.1): For $n \geq 2$, if $s_{(n-1)} \in \mathbb{P}_{n-1}$, let $P'_n(s_{(n-1)})$ denote a relatively compact base for the cone $\Gamma(P_n(s_{(n-1)}))$. To avoid treating the case for n = 1 separately, the following conventions are adopted. For n = 1, the statement "$s_{(n-1)} \in \mathbb{P}_{n-1}$ (or $S_{(n-1)}$) and ${}^i\delta_{n-1} \in {}^i\Delta_{n-1} \cap (H_{(n-1)}[s_{(n-1)}] \times A_{n-1})$" is vacuous, while the symbolism "$\alpha^2_{n-1}({}^i\delta_{n-1})$" or "$\alpha^2_{n-1}({}^ih_{(n)})$" stands for ${}^i\omega_1 + {}^ib_0$. Similarly, for n = 1 the symbolism "$P'_n(s_{(n-1)})$" or "$P_n(s_{(n-1)})$" means $\mathbb{P}_1$ (except where noted in the proof of Theorem (4.2.6) in Section 4.3).

Definition (4.2.2): Suppose $s_{(n-1)} \in S_{(n-1)}$, and $^i\delta_{n-1} \in {}^i\Delta_{n-1} \cap (H_{(n-1)}[s_{(n-1)}] \times A_{n-1})$, each $i \in \mathcal{L}$. A temporary competitive equilibrium (TCE) for $\mathcal{E}$ at date n relative to the histories $^i\delta_{n-1}$, $i \in \mathcal{L}$, is an I + 1 -tuple $(s_n^*, {}^1a_n^*, \ldots, {}^Ia_n^*)$ such that

$$(4.2.3)(i) \qquad s_n^* \in P_n(s_{(n-1)})$$

$$(4.2.3)(ii) \qquad {}^ia_n^* \in {}^id_n({}^ih_{(n)}^*), \ i \in \mathcal{L},$$

$$(4.2.3)(iii) \qquad \Sigma_{i\in\mathcal{L}}\, {}^ia_n^* = \Sigma_{i\in\mathcal{L}}\, (\alpha_{n-1}^2({}^ih_{(n)}^*),\ {}^i\omega_{n+1}),$$

where ${}^ih_{(n)}^* = ({}^i\delta_{n-1}, s_n^*)({}^ih_{(1)}^* = s_1^*)$, $i \in \mathcal{L}$.

Remarks (4.2.4)(i): The restriction in (4.2.3)(i) serves to limit the scope of the concept of a TCE. We have no knowledge about the existence of equilibria that do not satisfy this restriction, yet there are concepts and results involving the class of equilibria that do satisfy this restriction. In addition, one can interpret (4.2.3)(i) for $n > 1$ as requiring that opinions be fulfilled in a weak sense at equilibrium, i.e., the equilibrium price system must be a common price forecast at the relevant history. For each $n \geq 1$, (under (A.4)) (4.2.3)(i) also requires that the futures component of the equilibrium price system s_n^* be a common spot price forecast for date n + 1 at the history $(s_{(n-1)}, s_n^*)(= s_1^*$, if $n = 1)$.

(4.2.4)(ii): Condition (4.2.3)(ii) states that an equilibrium action ${}^ia_n^*$ maximizes agent i's conditional expected utility given the history ${}^ih_{(n)}^*$ subject to the appropriate budget and planning constraints at that history. Trivially then this condition entails that every agent is solvent at s_n^*.

(4.2.4)(iii): Condition (4.2.3)(iii) is the market clearing condition. The first ℓ equalities require that aggregate demand on current spot markets, $\Sigma_{i\in\mathcal{L}}\,\alpha_n^1({}^ia_n^*)$, equal aggregate supply of commodities at date n after settlement of futures contracts made at date n - 1. Total futures contracts of agent i at date n - 1 is the vector ${}^{i}b_{n-1} = {}^{i}f_{n-1} - {}^{i}\omega_n$, where ${}^{i}f_{n-1} = \alpha^2_{n-1}({}^{i}h^*_{(n)})$. Aggregate supply after settlement of date n - 1 contracts is simply $\Sigma_{i\in\mathcal{I}}({}^{i}\omega_n + {}^{i}b_{n-1}) = \Sigma_{i\in\mathcal{I}}\alpha^2_{n-1}({}^{i}h^*_{(n)})$. The second ℓ equalities in (4.2.3)(iii) require that date n futures markets clear, i.e., that $\Sigma_{i\in\mathcal{I}}{}^{i}b^*_n = \Sigma_{i\in\mathcal{I}}({}^{i}f^*_n - {}^{i}\omega_{n+1}) = 0$, where ${}^{i}f^*_n = \alpha^2_n({}^{i}a^*_n)$.

<u>Definition</u> (4.2.5): For each n, let $H^*_n = {}^{1}H^*_n \times\ldots\times {}^{I}H^*_n$, with generic element h_n, and define the correspondence $\varphi_n : H^*_n \Rightarrow \mathbb{R}^{2\ell}$ by $\varphi_n(h_n) = \Sigma_{i\in\mathcal{I}}({}^{i}d_n({}^{i}h_{(n)}) - (\alpha^2_{n-1}({}^{i}h_{(n)}), {}^{i}\omega_{n+1}))$, where $h_n = ({}^{1}h_{(n)}, \ldots, {}^{I}h_{(n)})$. Then φ_n is the <u>aggregate excess demand</u> correspondence for date n. If $s_{(n-1)} \in S_{(n-1)}$ and ${}^{i}\delta_{n-1} \in {}^{i}\Delta_{n-1} \cap (H_{(n-1)}[s_{(n-1)}] \times A_{n-1})$ for each $i \in \mathcal{I}$, write $\varphi_n({}^{\mathcal{I}}\delta_{n-1}, s_n)$ for $\varphi_n(({}^{1}\delta_{n-1}, s_n), \ldots, ({}^{I}\delta_{n-1}, s_n))$.

The following is the major existence result. Its proof is given in Section 4.3.

<u>Theorem</u> (4.2.6): If $s_{(n-1)} \in \mathbb{P}_{n-1}$ and ${}^{i}\delta_{n-1} \in {}^{i}\Delta_{n-1} \cap (H_{(n-1)}[s_{(n-1)}] \times A_{n-1})$, for each $i \in \mathcal{I}$, and if $\Sigma_{i\in\mathcal{I}}\alpha^2_{n-1}({}^{i}\delta_{n-1}) = \bar{\omega}_n$, then there exists s^*_n such that $s^*_n \in P'_n(s_{(n-1)})$ and $0 \in \varphi_n({}^{\mathcal{I}}\delta_{n-1}, s^*_n)$.

<u>Definition</u> (4.2.7). A sequence $\{(s^*_n, {}^{1}a^*_n, \ldots, {}^{I}a^*_n)\}_{n\in\mathbb{N}}$ is an <u>equilibrium path</u> for E if $(s^*_1, {}^{1}a^*_1, \ldots, {}^{I}a^*_1)$ is a TCE for E at date 1, and for n > 1, $(s^*_n, {}^{1}a^*_n, \ldots, {}^{I}a^*_n)$ is a TCE for E at date n relative to the histories ${}^{i}\delta^*_{n-1} = (s^*_1, {}^{i}a^*_i, \ldots, {}^{i}a^*_{n-1})$, $i \in \mathcal{I}$. Thus an equilibrium path is a sequence of temporary equilibria in which the economy is in temporary equilibrium at each date relative to the past history of temporary equilibria.

<u>Corollary</u> (4.2.8): There exists an equilibrium path for E.

Proof: The crucial hypothesis of Theorem (4.2.6) is that for the relevant histories, $\Sigma_{i\in\mathcal{A}}\alpha^2_{n-1}({}^i\delta_{n-1}) = \bar{\omega}_n$, i.e., that the futures markets cleared at date n - 1. For n = 1, this condition holds by (A.0). By the theorem, there exists a TCE $(s^*_1, {}^1a^*_1, \ldots, {}^Ia^*_1)$ for E at date 1, and $s^*_1 \in \mathbb{P}_1$. By (4.2.3)(iii), s^*_1 and ${}^i\delta^*_1 = (s^*_1, {}^ia^*_1)$, $i \in \mathcal{A}$, satisfy the hypotheses of the theorem. Hence there exists a TCE $(s^*_2, {}^1a^*_2, \ldots, {}^Ia^*_2)$ for E at date 2 relative to the histories ${}^i\delta^*_1$, $i \in \mathcal{A}$, and $s^*_2 \in P_2(s^*_1)$. Thus $s^*_{(2)} = (s^*_1, s^*_2) \in \mathbb{P}_2$ and ${}^i\delta^*_2 = ({}^i\delta^*_1, s^*_2, {}^ia^*_2)$, $i \in \mathcal{A}$, satisfy the hypotheses of the theorem, etc. □

For an equilibrium path to make economic sense, it must be realizable from the perspective of the model of intertemporal choice for the agents. There must exist a solution to the agent's expected utility maximization problem as formulated in [8] that actually selects the successive equilibrium actions at the equilibrium prices. It follows from [8: Lemma (5.2.3)(i) - (iii)] that for the typical agent there exists a sequence $\pi^* = \{\pi^*_n\}_{n\in\mathbb{N}}$ of measurable functions $\pi^*_n : H_{(n)} \to A_n$ such that π^*_n selects from D_n on dom D_n and from d_n on H^*_n. It is shown in [8: Theorem (5.2.8)] that any such sequence π^* is a solution to the agent's intertemporal choice problem.

<u>Definition</u> (4.2.9): An equilibrium path $\{(s^*_n, {}^1a^*_n, \ldots, {}^Ia^*_n)\}_{n\in\mathbb{N}}$ is said to be <u>realizable</u> if there is an I-tuple $({}^1\pi^*, \ldots, {}^I\pi^*)$, where ${}^i\pi^* = \{{}^i\pi^*_n\}_{n\in\mathbb{N}}$ is a sequcne of functions satisfying [8: Lemma (5.2.3)(i) - (iii)] and ${}^i\pi^*_n({}^i\delta^*_{n-1}, s^*_n) = {}^ia^*_n$, for each $i \in \mathcal{A}$, $n \in \mathbb{N}$, and for ${}^i\delta^*_{n-1}$ as in Definition (4.2.7). In this case, the equilibrium path is said to be <u>realizable by</u> $({}^1\pi^*, \ldots, {}^I\pi^*)$. An I + 1-tuple $(s^*, {}^1\pi^*, \ldots, {}^I\pi^*)$ is a <u>competitive equilibrium</u> (or just equilibrium) for E if $s^* = \{s^*_n\}_{n\in\mathbb{N}}$ is the sequence of price systems of some equilibrium path for E that is realizable by $({}^1\pi^*, \ldots, {}^I\pi^*)$.

While it is important to point out that an equilibrium path, to make sense, should be realizable, the existence of an equilibrium for E is a trivial issue in this model.

Theorem (4.2.10): Every equilibrium path for E is realizable and hence corresponds to an equilibrium for E.

Proof: Follows directly from the second statement of [8: Lemma (5.2.3)]. □

4.3 Proof of Theorem (4.2.6)

The strategy in proving Theorem (4.2.6) is essentially the same as in [5: Section 4]. The desired behavior of demand and aggregate excess demand is obtained in three lemmas, the first two of which are analogs respectively of [5: Lemma 4.3 and Theorem 4.4]. If $i \in \mathcal{A}$ and $h_{(n)} \in \text{dom } {}^i d_n$, let $\inf {}^i d_n(h_{(n)}) = \inf \{\|a_n\| : a_n \in {}^i d_n(h_{(n)})\}$. Also if $h_n \in H_n^*$, let $\inf \varphi_n(h_n) = \inf \{\|z\| : z \in \varphi_n(h_n)\}$.

Lemma (4.3.1): Suppose that $s_{(n-1)} \in \mathbb{P}_{n-1}$ and ${}^i\delta_{n-1} \in {}^i\Delta_{n-1} \cap (H_{(n-1)}[s_{(n-1)}] \times A_{n-1})$, $i \in \mathcal{A}$. If $\Sigma_{i\in\mathcal{A}} \alpha^2_{n-1}({}^i\delta_{n-1}) = \bar{\omega}_n$ and if $\{s_n^k\}_{k\in\mathbb{N}} \subseteq P_n'(s_{(n-1)})$ with $\lim_k s_n^k = s_n \notin P_n'(s_{(n-1)})$, then for some $\underline{i} \in \mathcal{A}$, $\lim_k \inf {}^{\underline{i}}d_n({}^{\underline{i}}\delta_{n-1}, s_n^k) = \infty$.

Proof: Let $s_{(n-1)}$, ${}^i\delta_{n-1}$, $\{s_n^k\}_{k\in\mathbb{N}}$, and s_n be as hypothesized, and let ${}^i h_{(n)} = ({}^i\delta_{n-1}, s_n)$, ${}^i h_{(n)}^k = ({}^i\delta_{n-1}, s_n^k)$. By Corollary (3.2.3), $\alpha_{n-1}({}^i\delta_{n-1}) \in \mathbb{R}_+^{2\ell}$, each i. By definition of $P_n(s_{(n-1)})$, ${}^i h_{(n)}^k \in {}^i H_n^*$ and hence ${}^i d_n({}^i h_{(n)}^k) \neq \emptyset$, for each i and k. Let $\varepsilon > 0$ be given and choose ${}^i a_n^k \in {}^i d_n({}^i h_{(n)}^k)$ so that $\|{}^i a_n^k\| < \inf {}^i d_n({}^i h_{(n)}^k) + \varepsilon$.

Since $s_n \in \text{cl} P_n'(s_{(n-1)})$ but $s_n \notin P_n'(s_{(n-1)})$, it follows that

$s_n \notin \Gamma(P_n(s_{(n-1)}))$ (as in the proof of Lemma (3.3.4)). If there is no i such that $\|{}^i a_n^k\| \to \infty$, then we may assume (by passing to a subsequence if necessary) that ${}^i a_n^k \to {}^i a_n \in A_n$, each $i \in \mathcal{A}$. If $\xi_n^2(s_n) \in \mathbb{R}_{++}^{\ell}$, then ${}^i r_n({}^i h_{(n)}) > 0$ and ${}^i D_n$ is l.h.c. at ${}^i h_{(n)}$, for each $i \in \mathcal{A}$ [8: Theorem (3.2.6)]. By [8: Lemma (5.3.4)] it follows that ${}^i a_n \in {}^i d_n({}^i h_{(n)})$, $i \in \mathcal{A}$. But by (A.7), $s_n \notin \cap_{i\in\mathcal{A}} {}^i Q_n(s_{(n-1)})$ and hence for some $i \in \mathcal{A}$, ${}^i d_n({}^i h_{(n)}) = \emptyset$ [8: Theorem (5.3.5)], a contradiction. If $\xi_n^2(s_n) \in \mathbb{R}_+^{\ell} \setminus \mathbb{R}_{++}^{\ell}$, then by definition $s_n \notin {}^i Q_n(s_{(n-1)})$ for every $i \in \mathcal{A}$. Since $s_n \in \mathrm{cl}P_n'(s_{(n-1)})$, $s_n \geq 0$ by Lemma (3.3.4). Consequently (by (A.0) or (A.1)) $s_n \cdot (\bar{\omega}_n, \bar{\omega}_{n+1}) = \Sigma_{i\in\mathcal{A}} {}^i r_n({}^i h_{(n)}) > 0$. Thus for some $i \in \mathcal{A}$, ${}^i r_n({}^i h_{(n)}) > 0$, implying as above that ${}^i a_n \in {}^i d_n({}^i h_{(n)})$ for this i, and the same contradiction obtains. □

A similar result obtains for aggregate excess demand as a consequence of this lemma and Corollary (3.2.3).

<u>Lemma</u> (4.3.2): Suppose that $s_{(n-1)} \in \mathbb{P}_{n-1}$ and ${}^i\delta_{n-1} \in {}^i\Delta_{n-1} \cap (H_{(n-1)}[s_{(n-1)}] \times A_{n-1})$, $i \in \mathcal{A}$. If $\Sigma_{i\in\mathcal{A}} \alpha_{n-1}^2({}^i\delta_{n-1}) = \bar{\omega}_n$ and if $\{s_n^k\}_{k\in\mathbb{N}} \subseteq P_n'(s_{(n-1)})$ with $\lim_k s_n^k = s_n \notin P_n'(s_{(n-1)})$, then $\lim_k \inf \varphi_n({}^{\mathcal{A}}\delta_{n-1}, s_n^k) = \infty$.

Proof: Let $s_{(n-1)}$, ${}^i\delta_{n-1}$, $\{s_n^k\}_{k\in\mathbb{N}}$, and s_n be as hypothesized, and let ${}^i h_{(n)}^k = ({}^i\delta_{n-1}, s_n^k)$, each i and k. Given $\varepsilon > 0$, choose $z^k \in \varphi_n({}^{\mathcal{A}}\delta_{n-1}, s_n^k)$ such that $\|z^k\| \leq \inf \varphi_n({}^{\mathcal{A}}\delta_{n-1}, s_n^k) + \varepsilon$, and choose ${}^i a_n^k \in {}^i d_n({}^i h_{(n)}^k)$ so that $z^k = (\Sigma_{i\in\mathcal{A}} {}^i a_n^k) - (\bar{\omega}_n, \bar{\omega}_{n+1})$. Then $|\|\Sigma_{i\in\mathcal{A}} {}^i a_n^k\| - \|(\bar{\omega}_n, \bar{\omega}_{n+1})\|| \leq \|z^k\|$. By Corollary (3.2.3), ${}^i a_n^k \in \mathbb{R}_+^{2\ell}$, each i and k, and by Lemma (4.3.1), $\|{}^{\underline{i}} a_n^k\| \to \infty$ for some $\underline{i} \in \mathcal{A}$. Thus $0 \leqq {}^{\underline{i}} a_n^k \leqq \Sigma_{i\in\mathcal{A}} {}^i a_n^k$, and the result follows. □

Recall Definition (4.2.5) and endow the set H_n^* with the product topology.

Lemma (4.3.3): φ_n is non-empty, compact, and convex valued and u.h.c. on H_n^*.

Proof: Follows directly from [8: Lemmas (5.1.6), (5.3.1)] and standard arguments (e.g., [6: Propositions 4 and 5, p. 25]. □

Let $s_{(n-1)}$ and ${}^i\delta_{n-1}$, $i \in \mathcal{A}$, be as hypothesized in Theorem (4.2.6). It follows from [8: Lemma (4.4.3)] that there exists a sequence $\{P_n^k\}_{k\in\mathbb{N}}$ of compact convex sets such that $\emptyset \neq \text{int } P_n^1 \subseteq P_n^2 \subseteq \ldots$ and $P_n(s_{(n-1)}) = \cup_{k\in\mathbb{N}} P_n^k$. Let φ_n^k denote the restriction of $\varphi_n({}^{\mathcal{A}}\delta_{n-1}, \cdot)$ to P_n^k. Without loss of generality we may assume that $\hat{P}_n^k \equiv P_n^k \cap P_n'(s_{(n-1)}) \neq \emptyset$, for each k. (For the case n = 1, use $\lambda\Omega^2 \cap \mathbb{P}_1$ for $P_n'(s_{(n-1)})$, where Ω^2 is the unit simplex in $\mathbb{R}^{2\ell}$ and λ is a positive scalar such that $\lambda\Omega^2 \cap \mathbb{P}_1 \neq \emptyset$.). Clearly $\hat{P}_n^k$ is compact and convex and $P_n'(s_{(n-1)}) = \cup_{k\in\mathbb{N}} \hat{P}_n^k$. Let Z^k denote the range of φ_n^k on $\hat{P}_n^k$. It follows from Lemma (4.3.3) that Z^k is compact. By [8: Corollary (5.3.3)], for each i and k and every $s_n \in \hat{P}_n^k$, $s_n \cdot {}^i d_n({}^i\delta_{n-1}, s_n) = s_n \cdot (\alpha^2_{n-1}({}^i\delta_{n-1}), {}^i\omega_{n+1})$, and summing on i gives Walras law $s_n \cdot \varphi_n^k(s_n) = 0$. It follows from [3] that for each k there exists $s_n^k \in \hat{P}_n^k$ and $z^k \in \varphi_n^k(s_n^k)$ such that $s_n \cdot z^k \leq s_n^k \cdot z^k = 0$ for all $s_n \in \hat{P}_n^k$.

Since $\{s_n^k\}_{k\in\mathbb{N}} \subseteq P_n'(s_{(n-1)})$, there is a subsequence of this sequence converging to some $s_n^* \in \text{cl}P_n'(s_{(n-1)})$. For convenience use the same index k for this subsequence. If $s_n^* \in P_n'(s_{(n-1)})$, then by Lemma (4.3.3), there is a subsequence of $\{z^k\}_{k\in\mathbb{N}}$ converging to some $z^* \in \varphi_n({}^{\mathcal{A}}\delta_{n-1}, s_n^*)$. Clearly $s_n^* \cdot z^* = 0$, and since $s_n^* \in P_n'(s_{(n-1)}) = \text{ri}P_n'(s_{(n-1)}) \subseteq \mathbb{R}^{2\ell}_{++}$, it follows by a standard argument that $z^* = 0$.

To prove that $s_n^* \in P_n'(s_{(n-1)})$, it suffices by Lemma (4.3.2) to show that the sequence $\{z^k\}_{k\in\mathbb{N}}$ is bounded. For each i and k let ${}^i a_n^k \in {}^i d_n({}^i\delta_{n-1}, s_n^k)$ be such that $z^k = (\Sigma_{i\in\mathcal{A}} {}^i a_n^k) - (\bar{\omega}_n, \bar{\omega}_{n+1})$ (recall that $\bar{\omega}_n = \Sigma_{i\in\mathcal{A}} \alpha^2_{n-1}({}^i\delta_{n-1})$, by assumption). By Corollary (3.2.3), ${}^{i}a_{n}^{k} \in \mathbb{R}_{+}^{2\ell}$, each i and k. Let $s_n \in P_n'(s_{(n-1)})$ be arbitrary. Then for k sufficiently large, $s_n \cdot z^k \leq 0$, which implies that $0 \leq s_n \cdot \Sigma_{i\in\mathcal{A}} {}^{i}a_{n}^{k} \leq s_n \cdot (\bar{\omega}_n, \bar{\omega}_{n+1})$. Since $s_n \in \mathbb{R}_{++}^{2\ell}$, it follows that $\Sigma_{i\in\mathcal{A}} {}^{i}a_{n}^{k}$ is bounded, and hence so is $\{z^k\}_{k\in\mathbb{N}}$. □

5. Conclusions

The results of this paper resolve in a positive way the issue of viability of a purely competitive exchange mechanism in the context of sequential trading in spot and futures markets with no institutional arrangements for handling bankruptcy. The backbone of these results consists of the four assumptions (A.3)-(A.7) made on agents' opinions. Each of these assumptions has been discussed and interpreted as they were presented here and in [8]. The informal remarks in this section will be confined to the question of consistency of these assumptions and to some related points regarding their restrictive nature.

Fortunately, examples that are trivial in one sense suffice to verify consistency of (A.3)-(A.7). Let ${}^{i}q_n\colon S_{(n)} \to \mathcal{P}(S_{n+1})$ be a continuous function chosen so that $\operatorname{supp} {}^{i}q_n(s_{(n)}) = \mathbb{R}^{2\ell}_{+}$ and ${}^{i}q_n(\mathbb{R}^{2\ell}_{++} | s_{(n)}) = 1$, for all i, n, and $s_{(n)}$. In this case, ${}^{i}Q_1 \equiv {}^{i}Q_n(s_{(n-1)}) \equiv \mathbb{R}^{2\ell}_{++}$ for each i and (A.3) and (A.4) hold. It follows also that $\mathbb{P}_1 \equiv P_n(s_{(n-1)}) \equiv \mathbb{R}^{2\ell}_{++}$, and (A.5), (A.6), and (A.7) hold as well, verifying the consistency of (A.3)-(A.7).

The sense in which this example is trivial has to do with the constancy of the supports of the ${}^{i}q_n$. The measures ${}^{i}q_n(s_{(n)})$ may vary in a non-trivial (but continuous) fashion with the price history $s_{(n)}$ and may vary from agent to agent, but the support ${}^{i}\sigma_n(s_{(n)})$ is constant in both $s_{(n)}$ and i (and in n).

Actually, one can do a little better than this without much effort. Let ${}^{i}\sigma_n$ be a continuous (u.h.c. and l.h.c.) correspondence on $s_{(n)}$ with compact and convex values in $\mathbb{R}^{2\ell}_{+}$ such that $\Gamma(\operatorname{int} {}^{i}\sigma_n(s_{(n)})) = \mathbb{R}^{2\ell}_{++}$ for every i, n, and $s_{(n)}$. Also let ${}^{i}q_n$ be a continuous measure valued function that generates ${}^{i}\sigma_n$, e.g., ${}^{i}q_n(s_{(n)})$ might be normalized Lebesgue measure on ${}^{i}\sigma_n(s_{(n)})$. Then again one has that ${}^{i}Q_1 \equiv {}^{i}Q_n(s_{(n-1)}) \equiv \mathbb{R}^{2\ell}_{++}$, and (A.3) and (A.4) hold for each i. If the ${}^{i}\sigma_n$ are chosen so that $\Gamma(\cap_{i\in\mathscr{A}}\mathrm{int}\,{}^{i}\sigma_n(s_{(n)})) = \mathbb{R}^{2\ell}_{++}$ for each n and $s_{(n)}$, then (A.5)-(A.7) hold as well.

In this example, the supports ${}^{i}\sigma_n(s_{(n)})$ can vary both in $s_{(n)}$ and across agents, but essentially only in terms of changes in scale, which may be interpreted loosely as price level (in a non-monetary sense) variability. In contrast, there is no variability of the ${}^{i}\sigma_n$ in terms of relative prices and the range of relative prices is maximal in the sense that for every i, n, and $s_{(n)}$, ${}^{i}\sigma_n(s_{(n)})$ must contain a compact base for the cone $\mathbb{R}^{2\ell}_{+}$. If (A.7) is to hold as well, the ${}^{i}\sigma_n$, $i \in \mathscr{A}$, must share a compact base for $\mathbb{R}^{2\ell}_{+}$ at $s_{(n)}$. Equivalently, all positive relative price systems (identified, e.g., with points of $\mathrm{ri}\Omega^2$) are common relative price forecasts for $n + 1$ at $s_{(n)}$.

One can obtain a sense of the restrictive nature of the assumptions (A.3)-(A.7) by trying to construct an example sastisfying these assumptions that exhibits a limited (as opposed to maximal) range of relative prices or relative spot prices. In doing so, one is frustrated by the basic implication of these assumptions, namely Theorem (3.2.2). It is clear from this result that one must specify the ${}^{i}\sigma_n$ in such a way that for important price histories $s_{(n)}$, one has that $\Omega^1 \times \{0\} \subseteq {}^{i}\bar{\sigma}_n(s_{(n)})$ for each i. One can show that for such $s_{(n)}$, $\Gamma(\cap_{i\in\mathscr{A}}\xi^1_{n+1}(\mathrm{int}\ {}^{i}\sigma_n(s_{(n)}))) = \mathbb{R}^{\ell}_{++}$. It follows that for these important price histories the range of relative spot prices is maximal. Every positive relative spot price system (every point in $\mathrm{ri}\Omega^1$) is a common spot price forecast (up to agent specific changes in scale) for $n + 1$ at $s_{(n)}$.

It is also clear, however, that what constitutes the important price histories at any date $n > 1$, the set $\mathbb{P}_n$, depends on the specification of the ${}^{i}\sigma_k$, $k = 1,\ldots,n - 1$. One might attempt to avoid this dependence by assuming that for a given history $s_{(n-1)}$, ${}^{i}\mathscr{J}_n(s_{(n-1)}) = \mathbb{R}^{2\ell}_{++}$, where ${}^{i}\mathscr{J}_n(s_{(n-1)})$ is the

$s_{(n-1)}$ - section of the set defined in (3.2.1). In words the assumption says that for every positive date n price system s_n, every positive relative spot price system is a spot price forecast (up to scale change) for agent i at $(s_{(n-1)}, s_n)$, i.e., $\Gamma(\xi^1_{n+1}(\text{int } {}^i\sigma_n(s_{(n-1)}, s_n))) = \mathbb{R}^{\ell}_{++}$. But by definition, this entails that ${}^iQ_n(s_{(n-1)}) = \mathbb{R}^{2\ell}_{++}$. If this assumption is made for each i, then (A.7) implies that $\Gamma(\cap_{i\in\mathcal{A}} \text{int } {}^i\sigma_{n-1}(s_{(n-1)})) = \mathbb{R}^{2\ell}_{++}$, the case of the previous example for this n and $s_{(n-1)}$.

The maximal range of relative spot price forecasts for important histories should come as no surprise. It is precisely this feature that is needed to avoid the possibility of bankruptcy (Corollary (3.2.3)) in the absence of institutional arrangements for this contingency.

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FOOTNOTES

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2. C.f. [8: Section 1.1] for a discussion and review of the literature.
3. Roughly speaking, a price forecast for an agent is a price system that is possible in the subjective opinion of the agent. The precise meaning of the term "Price forecast" employed in this work is slightly more restrictive. See [8: Definition (2.3.9)] and Section 2.2 below.
4. Bankruptcy here is synonomous with negative net worth. A market opportunity based definition of bankruptcy, which is somewhat less arbitrary, is discussed in [8: Section 3.1].
5. $\mathbb{R}$ denotes the set of real numbers, $\mathbb{R}^m$ the set of ordered m-tuples of real numbers, $m \in \mathbb{N}$, and $\mathbb{R}^\infty$ the set of sequences of real numbers. If $x = (x_1,\dots,x_m) \in \mathbb{R}^m$ and $y = (y_1,\dots,y_m) \in \mathbb{R}^m$, then $x \leqq y (y \geqq x)$ means $x_i \leq y_i$, $i = 1,\dots,m$, $x \leq y (y \geq x)$ means $x \leqq y$ and $x \neq y$, and $x < y (y > x)$ means $x_i < y_i$, $i = 1,\dots,m$. For $0 \in \mathbb{R}^m$, $\mathbb{R}^m_+ = \{x: x \in \mathbb{R}^m, x \geqq 0\}$, $\mathbb{R}^m_{+0} = \{x: x \in \mathbb{R}^m, x \geq 0\}$, and $\mathbb{R}^m_{++} = \{x: x \in \mathbb{R}^m, x > 0\}$. These relations and sets are defined analogously on $\mathbb{R}^\infty$. If $x \in \mathbb{R}^m$, $\|x\|$ denotes the Eculidean norm of x.
6. C.f. [8: Section 3.1] and Section 2.3 below for the sense of the term "net".
7. Products of measurable (topological) spaces are given the product σ-algebra (topology). For a topological space X, $\mathbb{B}(X)$ denotes the Borel subsets of X and $\mathcal{P}(X)$ the set of probability measures on $\mathbb{B}(X)$ endowed with the topology of weak convergence [2].

8. It is convenient to distinguish a relation from a function by use of the symbol " $\Rightarrow$" in the former case and " $\rightarrow$" in the latter case. All other notation and terminology follows [6: pp.4-5].

9. The notation "cl" and "int" denote, respectively, closure and interior relative to the ambient factor space of H in which the set lies. In this case, the space is S_{n+1}. The relative interior of a convex set is denoted by "ri".

10. C.f. also the discussion following [8: (3.1.6)].

11. This is really the thrust of [5: (4.1)]. Using the notation of [5], $\tilde{p}(p) \in \cap_{i \in \mathcal{I}} \text{int co}\, {}^{i}\sigma_n(p)$ for every $p \in P$. Except for the possible non-convexity of ${}^{i}\sigma_n(p)$ allowed in [5], in the terminology used here this says that $\tilde{p}(p)$ is a common price forecast for date 2 at the history p for each $p \in P$.

12. In the notation and context of [5] the quantifier would be "for every $p \in \Delta^{\ell}$".

13. The term "common spot price forecast" is a slight abuse of language, but the qualifying phase "up to agent specifice changes in scale" is cumbersome and this qualifier is dropped where convenient.

14. Clearly the hypothesis could be restricted to $(s_{(n)}, s_{n+1}) \in \mathbb{P}_n \times \text{cl}P_{n+1}(s_{(n)})$.

15. This supposition is a hypothesis that is implicit in all results of this section concerning the set $\tilde{P}^{*}_{n+1}$ and other objects denoted with an asterisk and all objects derived from these.

A NOTE ON "VALUATION EQUILIBRIUM AND PARETO OPTIMUM"

By

David C. Nachman

A NOTE ON "VALUATION EQUILIBRIUM AND PARETO OPTIMUM"[1]

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1. Introduction

Models of resource allocation under uncertainty and over time lead naturally to infinite dimensional commodity spaces. Debreu, in a now classic paper [1], extended the basic theorems of welfare economics to include a large class of infinite dimensional spaces, clarifiying in a number of ways the essentially algebraic and real analytic character of these theorems. Under convexity assumptions on preferences and consumption sets, Debreu showed that a valuation equilibrium in which no consumer is satiated is a Pareto optimum [1:Theorem 1]. This theorem relies only on the algebraic structure of the commodity space and algebraic (and order theoretic) properties of preferences and consumption sets, and does not depend at all on the dimensionality of these sets.

A nontrivial (continuous) converse of this result is that a Pareto optimum is a valuation equilibrium relative to a nontrivial (continuous) linear functional. Debreu obtained a continuous converse [1:Theorem 2 and Remark], when the commodity space is a topological vector space, under the additional assumptions that at least one consumer is nonsatiated, that preferences are continuous in a weak sense, which relies only on the analytical structure of the reals and not the topology of the commodity space, that the aggregate production set is convex, and that this set has an interior point or the commodity space is finite dimensional.

Both of these last assumptions impose dimensionality restrictions on the sets involved, the interior point assumption implying full dimensionality of the aggregate production set. The result relies on the Hahn-Banach theorem for the existence of the valuation functinal. If there is a prior reason to obtain continuity of this functional with respect to a given topology, then in the infinite dimensional case some interior point assumption is essential.

In general, however, it is possible to relativise such an assumption to an appropriate subspace of the commodity space and thereby free the theorem of dimensionality restrictions. In addition, if one is not concerned with continuity of the valuation functional with respect to some particular topology, then the theorem can be obtained with an algebraic analog of the interior point assumption. In this form, the converse theorem is an algebraic/real analytic result free of dimensionality restrictions and is a natural generalization of the finite dimensional result. In the finest locally convex topology for the commodity space the algebraic assumptions and topological assumptions coincide and every linear functional is continuous, as in the finite dimensional case.

The purpose of this note is to make these claims precise and to thereby clarify further the character of the converse theorem. The needed facts about vector spaces, internal points, etc., and the versions of the separation theorem used are presented in Section 2. The results on the converse theorem are presented in Section 3.

2. The Separation Theorem

2.1. Notation and Terminology. Except where noted, notation and terminology follow that in [6]. Throughout, E denotes a vector space over the real field R. If $B \subseteq E$, $B \neq \emptyset$, span (B) denotes the smallest subspace of E containing B, $L(B) = \text{span}(B - B)$, and $M(B)$ denotes the smallest linear manifold containing B. A point $b \in B$ is called an internal point of B if for each $x \in M(B)$ there exists $\varepsilon > 0$ such that $|\lambda| < \varepsilon$ implies $b + \lambda(x - b) \in B$. If E replaces $M(B)$ in this statement, the resulting definition is that of an algebraic interior point of B.[2] The set of algebraic interior points of B, the algebraic interior of B, is denoted by B^0.[3] The algebraic hull of B, denoted by B^a, is the set of points $x \in E$ for which there exists a point $b \in B$ such that $(b, x) \subseteq B$.[4]

The notion of internal point generalizes the concept of relative interior of a convex set to arbitrary dimensions. The key hypothesis of the Hahn-Banach theorem (aside from convexity and the axiom of choice) is the existence of an internal point. This hypothesis is superfluous in the finite dimensional case since every non-empty finite dimensional convex set has a non-empty relative interior and hence an internal point. This is no longer true for infinite dimensional convex sets [6:Example 1, p. 177], and the hypothesis is essential.

2.2. Theorem. If $B_1, B_2 \subseteq E$ are convex, B_1 has an internal point, $B_2 \subseteq M(B_1)$ and B_2 contains no internal point of B_1, then there exists a nontrivial linear function $f: E \to R$ such that $f(b_1) \geq (>)\ f(b_2)$ for all $b_1 \in B_1$ (b_1 an internal point of B_1) and all $b_2 \in B_2$.

Proof: Let b_1^0 be an internal point of B_1. Then $0 \in E$ is an internal point of $B_1 - b_1^0$ and $B_2 - b_1^0$ contains no internal points of $B_1 - b_1^0$. Since $0 \in B_1 - b_1^0$, span $(B_1 - b_1^0) = L(B_1 - b_1^0) = L(B_1) = M(B_1) - b_1^0$ by [6:§ 16.2.(3)]. By assumption $B_2 - b_1^0 \subseteq M(B_1) - b_1^0$. It follows from the Hahn-Banach theorem [6:§ 17.1.(3)] that there exists a nontrivial linear functional $f: M(B_1) - b_1^0 \to R$ and satisfying $f(b_1 - b_1^0) \geq (>) f(b_2 - b_1^0)$ for every $b_1 \in B_1$ (b_1 an internal point of B_1) and every $b_2 \in B_2$.[5] By [6:§ 9.2.(1)], f can be extended to a linear functional on all of E. Let f also denote such an extension. It follows as in the above inequality that $f(b_1) - f(b_1^0) = f(b_1 - b_1^0) \geq (>) f(b_2 - b_1^0) = f(b_2) - f(b_1^0)$, and the proof is complete.□

2.3. <u>Remark</u>. The hypothesis in 2.2. that $B_2 \subseteq M(B_1)$ is to ensure that the Hahn-Banach theorem can be applied to the appropriate subspace of E. In general, to apply this theorem to separate B_1 and B_2, the internal point assumption must be stated relative to a manifold (subspace) containing (translates of) both B_1 and B_2. In Section 3, B_2 will be a singleton and the hypothesis will hold.

The following assumption provides a useful sufficient condition for the sum of sets to have an internal point.

2.4. <u>Lemma</u>. If $B_1, B_2 \subseteq E$, then $M(B_1 + B_2) = M(B_1) + M(B_2) = M(B_1 + M(B_2))$. If b_i is an internal point of B_i, $i = 1, 2$, then $b_1 + b_2$ is an internal point of $B_1 + B_2$.

Proof: If $x \in M(B_1 + B_2)$, then $x = \Sigma_{k=1}^n \gamma_k (b_{1_k} + b_{2_k})$, for some $b_{ik} \in B_i$, $\gamma_k \in R$, $k = 1, \ldots, n$, $i = 1, 2$, with $\Sigma_{k=1}^n \gamma_k = 1$. But then

$x_i = \Sigma_{k=1}^{n} \gamma_k b_{ik} \in M(B_i)$, $i = 1, 2$, and $x = x_1 + x_2$. To go the other way, write $B_1 + B_2 = \cup_{b_1 \in B_1} b_1 + B_2$. Then $M(B_1 + B_2) = M(\cup_{b_1 \in B_1} b_1 + B_2) \supseteq M(b_1^* + B_2) = b_1^* + M(B_2)$ for each $b_1^* \in B$. It follows that $B_1 + M(B_2) = \cup_{b_1^* \in B_1} (b_1^* + M(B_2)) \subseteq M(B_1 + B_2)$ and hence $M(B_1 + M(B_2)) \subseteq M(B_1 + B_2)$. If $x \in M(B_1) + M(B_2)$, then $x = x_1 + x_2$, where $x_i \in M(B_i)$, $i = 1, 2$. Suppose $x_1 = \Sigma_{k=1}^{n} \gamma_k b_{1_k}$, $b_{1_k} \in B_1$, $\gamma_k \in R$, $\Sigma_{k=1}^{n} \gamma_k = 1$. Then $x = \Sigma_{k=1}^{n} \gamma_k (b_{1_k} + x_2) \in M(B_1 + M(B_2))$, proving the first statement.

To prove the second statement, let b_i^0 be an internal point of B_i, $i = 1, 2$. If $x \in M(B_1 + B_2)$, then by the first statement, $x = x_1 + x_2$, where $x_i \in M(B_i)$, $i = 1, 2$. There exists $\varepsilon_i > 0$ such that $|\lambda| < \varepsilon_i$ implies $b_i^0 + \lambda(x_i - b_i^0) \in B_i$, $i = 1, 2$. Let $\varepsilon = \min[\varepsilon_1, \varepsilon_2]$. Then $|\lambda| < \varepsilon$ implies that $b_i^0 + \lambda(x_i - b_i^0) \in B_i$, $i = 1, 2$, and hence $b_1^0 + b_2^0 + \lambda(x_1 + x_2 - (b_1^0 + b_2^0)) \in B_1 + B_2$, and the proof is complete.□

2.5. <u>Remark</u>. The converse of the second statement of 2.4. is in general false. For example, if $B_1^0 \neq \emptyset$, then $(B_1 + B_2)^0 \neq \emptyset$ for any non-empty set B_2. In particular, if $b_1 \in B_1^0$, then $b_1 + b_2 \in (B_1 + B_2)^0$ for any $b_2 \in B_2$, but B_2 need have no internal point.

2.6. <u>The Natural Topology</u>. A convex set $C \subseteq E$ such that $C^0 \neq \emptyset$ is called a <u>convex algebraic body</u> or <u>convex</u> α-<u>body</u> [6:p. 180]. The collection of convex α-bodies is a base for a topology $\mathcal{J}^0$ for E called the <u>natural topology</u> for E [5:Section 3].[6] This terminology is justified in several respects. The pair $(E, \mathcal{J}^0)$ is a locally convex topological vector space [5:(3.3)], and hence if E is finite dimensional, $(E, \mathcal{J}^0)$ is topologically isomorphic to Euclidean n-space for some integer n [6:$\mathcal{J}$ 15.5.(1)]. In general, $\mathcal{T}^0$ is the finest locally convex topology for E [6:$\mathcal{J}$ 18.5.(5)], every linear manifold in E is closed in $(E,\mathcal{T}^0)$ [5:(3.1)], and hence every linear functional on E is $\mathcal{T}^0$ continuous [6:$\mathcal{J}$ 15.9.(1)], as in the finite dimensional case. In particular, the linear functional of Theorem 2.2. is $\mathcal{T}^0$ continuous.

If $C \subseteq E$ is a convex set, then the $\mathcal{T}^0$ interior and the algebraic interior of C coincide [5:(2.1), (4.5)]. In addition, the set of internal points of a convex set C and the interior of C in the relative $\mathcal{T}^0$ topology for M(C) are the same. (If $0 \in E$ is an internal point of a convex set $C \subseteq E$, then $C = \hat{C} \cap M(C)$, where $\hat{C} = C + H$, H a complement of $M(C) = L(C)$ [6:$\mathcal{J}$ 7.3.(3)]. It is easily verified that $\hat{C}$ is convex and $0 \in \hat{C}^0$.).

Let $\mathcal{T}$ denote a topology for E for which the pair $(E,\mathcal{T})$ is a topological vector space. If H is a subspace of E, H in the relative topology from $\mathcal{T}$ is a topological vector space. If $B \subseteq E$, $B \neq \emptyset$, then M(B) in the relative topology is homeomorphic to L(B) via the mapping $x \to x + x_0$, $x_0 \in B$. The following is the topological analog of Theorem 2.2.

2.7. <u>Theorem</u>. Let $(E,\mathcal{T})$ be a topological vector space. If B_1, $B_2 \subseteq E$ are convex, B_1 has an interior point in $M(B_1)$, $B_2 \subseteq M(B_1)$ and B_2 contains no interior points (in $M(B_1)$) of B_1, then there exists a $\mathcal{T}$ continuous nontrivial linear functional $f:E \to R$ such that $f(b_1) \geq (>) f(b_2)$ for all $b_1 \in B_1$ (b_1 an interior point of B_1 in $M(B_1)$), and all $b_2 \in B_2$.

Proof: Follow the proof of Theorem 2.2. using [6:$\mathcal{J}$ 17.1.(4)] to obtain the existence of a continuous nontrivial linear functional $f:L(B_1) \to R$ and satisfying the desired inequalities. To extend f,[7] note first that there exists an open set $U \in \mathcal{T}$ with $0 \in U$ such that $|f(x)| < 1$ for all $x \in U \cap L(B_1)$. Let U' denote the absolute convex cover of $U \cap L(B_1)$ [6:$\mathcal{T}$ 16.1]. If $x \in U'$, then $x = \Sigma_{k=1}^{n}\alpha_k x_k$, $x_k \in U \cap L(B_1)$, $\alpha_k \in R$, $\Sigma_{k=1}^{n}|\alpha_k| \leq 1$, and $|f(x)| = |\Sigma_{k=1}^{n}\alpha_k\ f(x_k)| \leq \Sigma_{k=1}^{n}|\alpha_k|\ |f(x_k)| < 1$. Let H be a complementary subspace of $L(B_1)$ and let $\hat{U} = U' + H$. Then $\hat{U}$ is an open absolutely convex set in E with $0 \in \hat{U}$. By [6:$\mathcal{T}$ 16.4.(6), (7)], the Minkowsky functional $\hat{q}$ of $\hat{U}$ is a continuous semi-norm.

The set U' is open in $L(B_1)$ [6:$\mathcal{T}$ 16.1.(7)] and hence $L(B_1) = \text{span}\ (U')$. If $x \in U'$, $x \neq 0$, then $|f(x)| < 1$ and $|f(x)|^{-1} x \in \hat{U}^a \setminus \hat{U}$. By [6:$\mathcal{T}$ 16.4.(4)], $1 = \hat{q}(|f(x)|^{-1}x) = |f(x)|^{-1}\hat{q}(x)$, and hence $\hat{q}(x) = |f(x)|$. If $x \in L(B_1)$, then for some $\lambda > 0$, $\lambda x \in U'$. Then $|f(x)| = \lambda^{-1}|f(\lambda x)| = \lambda^{-1}\ \hat{q}(\lambda x) = \hat{q}(x)$. Thus on the subspace $L(B_1)$, the continuous linear functional f is dominated in absolute value by the semi-norm $\hat{q}$ which is defined and continuous on all of E. By an extension form of the Hahn-Banach theorem [6:$\mathcal{T}$ 17.3.(5)], f can be extended to a continuous linear functional on all of E.□

3. An Optimum is a Quasi-Equilibrium

3.1. <u>Notation and Assumptions</u>. With only minor changes, the notation in this section is the same as in [1]. The commodity space is denoted by E (instead of L), a vector space over R. No topological structure will be assumed for E.[8] In addition, X denotes the sum $\Sigma_{i=1}^{m} X_i$, $Z \equiv X - Y$, where $Y = \Sigma_{j=1}^{n} Y_j$, $\leq_i$ denotes the complete ordering of X_i (consumer i's preferences), $X_i(x_i) = \{x_i' \in X_i : x_i \leq_i x_i'\}$ and $\mathring{X}_i(x_i) = \{x_i' \in X_i : x_i <_i x_i'\}$, for each $x_i \in X_i$, and $x_i <_i x_i'$ means $x_i \leq_i x_i'$ and $x_i' \not\leq_i x_i$, $i = 1, \ldots, m$.

Assumptions I - IV of [1] are maintainted throughout this section.

3.2. Lemma. If $\mathring{X}_i(x_i) \neq \emptyset$, then $M(\mathring{X}_i(x_i)) = M(X_i(x_i))$.

Proof: Let $\mathring{M}_i = M(\mathring{X}_i(x_i))$ and $M_i = M(X_i(x_i))$. Clearly $\mathring{M}_i \subseteq M_i$. Suppose $x \in M_i$, $x = \Sigma_{k=1}^{r}\alpha_k x^k$, $x^k \in X_i(x_i)$, $\alpha_k \in R$, $\Sigma_{k=1}^{r}\alpha_k = 1$. Let $x^0 \in \mathring{X}_i(x_i)$, and let $\bar{k}$ be such that for $k = 1, \ldots, \bar{k}$, $x^k \in \mathring{X}_i(x_i)$, and $k = k + 1, \ldots, r$, $x^k \notin \mathring{X}_i(x_i)$, $0 \leq \bar{k} \leq r$. By I - III, for each $k = \bar{k} + 1, \ldots, r$, there exists $x^{0k} \in (x^0, x^k) \subseteq \mathring{X}_i(x_i)$ and $\lambda_k > 1$ such that $x^k = \lambda_k x^{0k} + (1 - \lambda_k)x^0$. Then $x = \Sigma_{k=1}^{\bar{k}}\alpha_k x^k + \Sigma_{k=\bar{k}+1}^{r}\alpha_k\lambda_k x^{0k} + \Sigma_{k=\bar{k}+1}^{r}\alpha_k(1 - \lambda_k)x^0$, $\Sigma_{k=1}^{\bar{k}}\alpha_k + \Sigma_{k=\bar{k}+1}^{r}\alpha_k(\lambda_k + (1 - \lambda_k))$ $= \Sigma_{k=1}^{r}\alpha_k = 1$, and $x \in \mathring{M}_i$.□

3.3. Definitions. A state $[(x_i^0), (y_j^0)]$ of the economy is called a quasi-equilibrium relative to a linear (valuation) functional ν on E if $[(x_i^0), (y_j^0), \nu]$ satisfies [1:(2.1), (5.1), (5.2)].[9] For a given state of the economy $[(x_i), (y_j)]$ and each $i' = 1, \ldots, m$, let $\mathring{Z}_{i'}[(x_i), (y_j)] = \mathring{X}_{i'}(x_{i'}) + \Sigma_{i\neq i'}X_i(x_i) - Y$, where any of the sets $X_i(x_i)$ may be empty; $i = 1,\ldots,m$.

3.4. Theorem. If $[(x_i^0), (y_j^0)]$ is a Pareto optimum, and for some $i' = 1, \ldots, m$, $x_{i'}^0$ is not a saturation point and $\mathring{Z}_{i'}[(x_i^0), (y_j^0)]$ has an internal point, then there is a nontrivial linear functional ν on E such that $[(x_i^0), (y_j^0)]$ is a quasi-equilibrium relative to ν.

Proof: Let $\wp \in E$ denote total resources. Then $\wp = x^0 - y^0$, where $x^0 = \Sigma_{i=1}^{m}x_i^0$, $y^0 = \Sigma_{j=1}^{n}y_j^0$. As in [1:Theorem 2], the set $\mathring{Z}_{i'} \equiv \mathring{Z}_{i'}[(x_i^0), (y_j^0)]$

is non-empty, convex, and $\mathcal{f} \notin \mathring{Z}_{i\cdot}$. By assumption $\mathring{Z}_{i\cdot}$ has an internal point. By Lemmas 2.4. and 3.3., $M(\mathring{Z}_{i\cdot}) = M(\mathring{X}_{i\cdot}(x^0_{i\cdot})) + \Sigma_{i\neq i\cdot} M(X_i(x^0_i)) - M(Y) = \Sigma^m_{i=1} M(X_i(x_i)) - \Sigma^n_{j=1} M(Y_j)$, and hence $\mathcal{f} \in M(\mathring{Z}_{i\cdot})$.

By Theorem 2.2., there is a nontrivial linear functional ν on E such that $\nu(z) \geq \nu(\mathcal{f})$ for every $z \in \mathring{Z}_{i\cdot}$. Arguing exactly as in [1:Theorem 2], it follows that $[(x^0_i), (y^0_j)]$ is a quasi-equilibrium relative to ν.□

3.5. Remark. Note that nowhere in the proof of Theorem 3.4. or in Debreu's proof [1:Theorem 2] is continuity of the linear functional ν required. The only place where a limiting argument is used is in obtaining [1:b), p.] and there linearity of ν suffices [1:(4.2)]. Of course the functional ν is $\mathcal{J}^0$-continuous, but this is synonymous with linearity. The topological dual of $(E, \mathcal{J}^0)$ is the algebraic dual of E.

3.6. Remark. Suppose $(E, \mathcal{J})$ is a topological vector space. Theorem 3.4. remains valid if the internal point assumption is replaced by the assumption that $\mathring{Z}_{i\cdot}[(x^0_i), (y^0_j)]$ has an interior point in the relative topology in $M(\mathring{Z}_i[(x^0_i), (y^0_j)])$. In this version, it follows from Theorem 2.7. that the functional ν can be taken to be $\mathcal{J}$-continuous, and the result is free of dimensionality restrictions.

3.7. Assumption. If E is finite dimensional, then under I - IV of [1], Y has an internal point if $Y \neq \emptyset$, and for each i and $x_i \in X_i$, $X_i(x_i)$ has an internal point as does $\mathring{X}_i(x_i)$ if these sets are non-empty. By (the back door of) Lemma 2.4., $\mathring{Z}_{i\cdot}[(x_i), (y_j)]$ has an internal point whenever this set is non-empty. The following assumption extends this trivial conclusion to the nontrivial infinite dimensional case.

V.′ For each j, Y_j has an internal point and for each i and $x_i \in X_i$, $X_i(x_i)$ has an internal point.

3.8. Lemma. Under I - III of [1] and V′ above, if $x_i \in X_i$, i = 1, ..., m, $y_j \in Y_j$, j = 1, ..., n, and if $\mathring{Z}_{i'}[(x_i), (y_j)] \neq \emptyset$, then $\mathring{Z}_{i'}[(x_i), (y_j)]$ has an internal point.

Proof: Follows at once from Lemma 2.4. and the following. □

3.9. Lemma. Under I - III of [1], if $X_i(x_i)$ has an internal point x_i^0 and $\mathring{X}_i(x_i) \neq \emptyset$, then $x_i^0 \in \mathring{X}_i(x_i)$ and x_i^0 is an internal point of $\mathring{X}_i(x_i)$.

Proof: Let x_i^0 be as hypothesized and let $x_i^{00} \in \mathring{X}_i(x_i)$. For $\lambda > 1$, let $x_\lambda = \lambda x_i^0 + (1-\lambda)x_i^{00}$. Then $x_\lambda \in M(X_i(x_i))$ and since x_i^0 is an internal point of $X_i(x_i)$, for some $\gamma \in (0, 1)$, $[x_i^0, \gamma x_\lambda + (1-\gamma)x_i^0] \subseteq X_i(x_i)$. By I - III, for every $\beta \in (0, 1)$, $\beta(\gamma x_\lambda + (1-\gamma)x_i^0) + (1-\beta)x_i^{00} \in \mathring{X}_i(x_i)$. Let $\beta = (\gamma\lambda + (1-\gamma))^{-1}$. Since $\lambda > 1$, $0 < \beta < 1$, and hence $\beta(\gamma x_\lambda + (1-\gamma)x_i^0) + (1-\beta)x_i^{00} = x_i^0 \in \mathring{X}_i(x_i)$. Also, by definition, if $x \in M(X_i(x_i))$, there is an $\varepsilon > 0$ such that $x_i^0 + \lambda'(x - x_i^0) \in X_i(x_i)$ for $|\lambda'| < \varepsilon$. Using I - III again, $X_i(x_i)$ may be replaced by $\mathring{X}_i(x_i)$ since $x_i^0 \in \mathring{X}_i(x_i)$.□

3.10. Remarks. Assumption V′ involves no dimensionality restrictions on the sets $X_i(x_i)$ and Y_j, but it does provide a sufficient condition for the internal point hypothesis of Theorem 3.4 to hold. In comparing V′ and [1 : V] we note that every infinite dimensional Banach space has a dense maximal subspace

H whose open half spaces H^+ and H^- have empty interiors (in the norm topology) but $(H^+)^0 = H^+$ and $(H^-)^0 = H^-$[5:p. 450].

3.11 Interpretation. The economic interpretation of V' is as follows. For a production set Y_j, the subspace $L(Y_j)$ can be thought of as the space of possible directions or changes in production plans. For if $d \in E$ and $y + d \in Y_j$ for some $y \in Y_j$, then $d \in L(Y_j)$. A direction $d \in L(Y_j)$ is feasible at $y \in Y_j$ if changes in y of arbitrarily small scale in the direction d are feasible, i.e., if for some $\varepsilon > 0$, $y + \lambda d \in Y_j$ for $0 \leq \lambda < \varepsilon$. All possible directions are feasible at $y \in Y_j$ if and only if y is an internal point of Y_j. Similarly, for a consumption set X_i and $x_i \in X_i$, the subspace $L(X_i(x_i))$ is the space of possible directions or changes in consumption plans at least as preferred as x_i that remain at least as preferred as x_i. Such a direction $d \in L(X_i(x_i))$ is feasible and preferred relative to x_i at $x \in X_i(x_i)$ if changes in x of arbitrarily small scale in the direction d are feasible and remain at least as preferred as x_i, i.e., if for some $\varepsilon > 0$, $x + \lambda d \in X_i(x_i)$ for $0 \leq \lambda < \varepsilon$. All possible such directions are feasible and preferred relative to x_i at $x \in X_i(x_i)$ if and only if x is an internal point of $X_i(x_i)$.

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FOOTNOTES

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2. The notion of internal point varies in the literature. The definition given here coincides with the notion of a point in the "intrinsic core" in [4:pp. 7-8]. The definition given in [6:p. 176] is ambiguous and could be taken to mean that b is an internal point of B if for each $x \in M(B)$, there exists $\lambda > 0$ such that $b + \lambda(x - b) \in B$. If B is convex, this interpretation is equivalent to the definition in the text and it is in the convex case where the notion is used in [6]. (That these two are not equivalent when B is not required to be convex can be seen by taking $E = R$ and B the set $\{-1, 0, 1\}$. Clearly 0 is an internal point in the weaker sense above, but B contains no interval.) The concept of internal point used in [3:Definition 7, p. 410] and in [7:p.204] and the concept of a point in the "core" of a set in [5:p. 445] all coincide with that of an algebraic interior point.

3. The algebraic interior of a set is synonymous with its "core" or "algebraic core" in [4], [5]. Subject to the ambiguity of footnote 1, the term "algebraic kernel" is used in [6:p. 177].

4. B^a is synonymous with "lin B" in [5:p. 448].

5. Köthe refers to [6: ʃʃ 17.1.(3)] as the algebraic form of the separation theorem and he reserves the name "Hahn-Banach theorem" for similar results [6:ʃʃ 17.2.]. The linear functional associated with the hyperplane of [6:ʃʃ 17.1.(3)] comes from [6:ʃʃ 15.9.(1)].

6. This typology is identified in [6:p. 214], but the term "natural topology"

has a different meaning [6:p. 300].

7. The extension of f is immediate from [6: ∬ 20.1.(1)] if $\mathcal{T}$ is locally convex.

8. The natural topology will perforce be involved. C.f. Section 2.6. above.

9. The term "quasi-equilibrium" is suggested by its use in [2] and by [1:Remark].